Degenerate Competing Three-Particle Systems

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at Lehigh University, March 2022, Seminar on Stochastic Processes

based on joint work [IK] with Ioannis Karatzas.

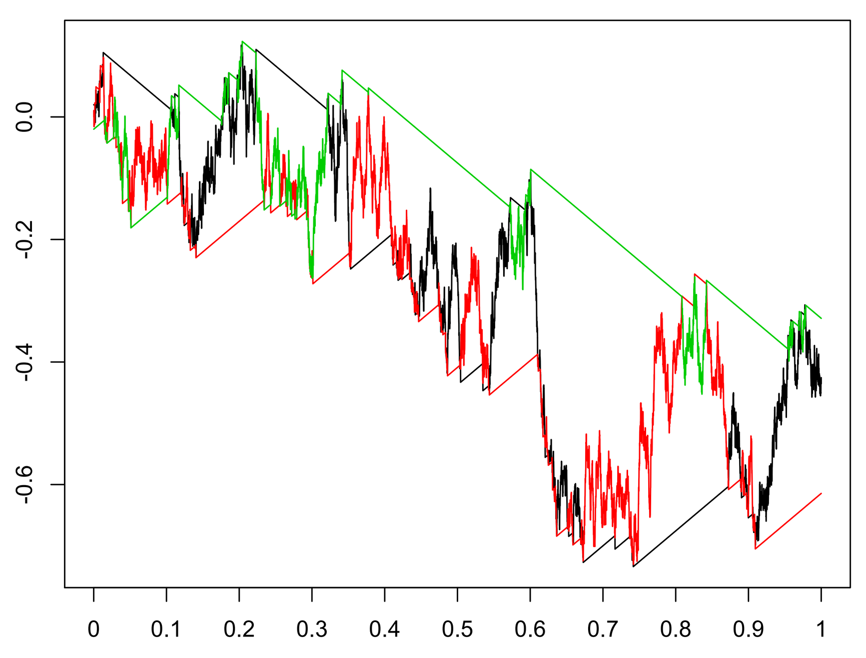

Fig 1. Simulated paths: $(X_1(\cdot),\ X_2(\cdot),\ X_3(\cdot))$ in (1).

We study systems of three interacting particles, in which drifts and variances are assigned by rank. These systems are degenerate: the variances corresponding to one or two ranks can vanish, so the corresponding ranked motions become ballistic rather than diffusive. Depending on which ranks are allowed to "go ballistic" the systems exhibit markedly different behavior, which we study here in some detail. Also studied are stability properties for the resulting planar process of gaps between successive ranks.

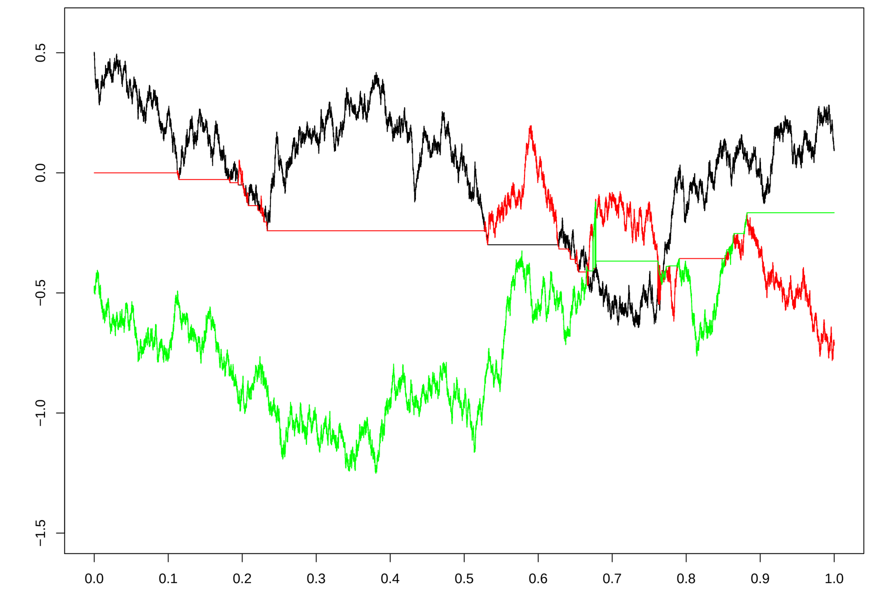

Fig 2. Simulated paths: $(X_1(\cdot),\ X_2(\cdot),\ X_3(\cdot))$ in (2).

1. Diffusion in middle

Given real numbers $\delta_1,\ \delta_2,\ \delta_3$ and $x_1 > x_2 > x_3$, we start by constructing a probability space $(\Omega, \mathfrak{F}, \mathbb{P})$ endowed with a right-continuous filtration $\mathbb{F} = \{\mathfrak{F}(t)\}_{0 \le t < \infty}$, to which are adapted three independent Brownian motions $B_1(\cdot),\ B_2(\cdot),\ B_3(\cdot)$, and three continuous processes $(X_1(\cdot),\ X_2(\cdot),\ X_3(\cdot))$ that satisfy the system for $i = 1, 2, 3$

$$X_i(\cdot) \,=\, x_i + \sum_{k=1}^{3} \delta_k \int_0^{\cdot} \mathbf{1}_{\{X_i(t) = R_k(t)\}}\, \mathrm{d}t + \int_0^{\cdot} \mathbf{1}_{\{X_i(t) = R_2(t)\}}\, \mathrm{d}B_i(t)\,, \quad (1)$$

$$\text{with the conditions} \quad \int_0^{\infty} \mathbf{1}_{\{R_k(t) = R_\ell(t)\}}\, \mathrm{d}t \,=\, 0\,, \quad \forall\ k < \ell,$$

$$\text{and} \quad \{t \in (0,\infty) :\ R_1(t) = R_3(t)\} \,=\, \emptyset$$

with probability one. Here we denote the descending order statistics by

$$\max_{j=1,2,3} X_j(t) =: R_1(t) \ge R_2(t) \ge R_3(t) := \min_{j=1,2,3} X_j(t)\,, \qquad t \in [0,\infty),$$

and adopt the convention of resolving ties always in favor of the lowest index i. The simulated sample paths are given in Fig 1.

Theorem 1. [Strong solution] The system of equations (1) admits a pathwise unique strong solution, satisfying the above conditions. With probability one, $R_1(\cdot) - R_3(\cdot) > 0$.

Proposition 2. [Positive recurrence] Under the stability conditions

$$2(\delta_3 - \delta_2) + (\delta_1 - \delta_2)^- > 0\,, \quad 2(\delta_2 - \delta_1) + (\delta_2 - \delta_3)^- > 0\,,$$

the gap process $(G(\cdot) := R_1(\cdot) - R_2(\cdot),\ H(\cdot) := R_2(\cdot) - R_3(\cdot))$ is positive-recurrent and has a unique invariant measure $\boldsymbol{\pi}$ with $\boldsymbol{\pi}((0,\infty)^2) = 1$.

- *"Two ballistic motions cannot squeeze a diffusive motion".*
- The Laplace transform of $\boldsymbol{\pi}$ and the basic adjoint relation that is satisfied by $\boldsymbol{\pi}$ are also discussed in [IK].

2. Ballistic motion in middle

We take up the "obverse" of the three-particle system in (1), by which for $i = 1, 2, 3$, we mean replacing the equations in (1) by

$$\begin{aligned} X_i(\cdot) \,&=\, x_i + \sum_{k=1}^{3} \delta_k \int_0^{\cdot} \mathbf{1}_{\{X_i(t) = R_k(t)\}}\, \mathrm{d}t \\ &\quad + \int_0^{\cdot} \big(\mathbf{1}_{\{X_i(t) = R_1(t)\}} + \mathbf{1}_{\{X_i(t) = R_3(t)\}}\big)\, \mathrm{d}B_i(t) \end{aligned} \qquad (2)$$

and replacing the conditions by

$$\int_0^{\infty} \mathbf{1}_{\{R_k(t) = R_\ell(t)\}}\, \mathrm{d}t \,=\, 0\,, \quad \forall\ k < \ell\,; \qquad L^{R_1 - R_3}(\cdot) \,\equiv\, 0\,, \qquad (3)$$

where $L^{\Xi}(\cdot)$ is the right local time at the origin of a continuous, nonnegative semimartingale of the form $\Xi(\cdot) = \Xi(0) + M(\cdot) + C(\cdot)$, with $M(\cdot)$ a continuous local martingale and $C(\cdot)$ a process of finite first variation on compact intervals:

$$L^{\Xi}(\cdot) \equiv L^{\Xi}(\cdot\,; 0) := \lim_{\varepsilon \downarrow 0} \frac{1}{2\varepsilon} \int_0^{\cdot} \mathbf{1}_{\{\Xi(t) < \varepsilon\}} \mathrm{d}\langle M \rangle(t)\,.$$

Theorem 3. On a filtered probability space $(\Omega, \mathcal{F}, \mathbb{P}), \mathbb{F} = \{\mathfrak{F}(t)\}_{t \ge 0}$ and with the process $X(\cdot)$, there exists a three-dimensional Brownian motion $B(\cdot) = (B_1(\cdot), B_2(\cdot), B_3(\cdot))'$ such that $(\Omega, \mathcal{F}, \mathbb{P}), \mathbb{F} = \{\mathfrak{F}(t)\}_{t \ge 0}\,, (X(\cdot), B(\cdot))$ is a weak solution for the system (2), (3).
This solution is unique in the sense of the probability distribution; thus, $X(\cdot)$ has the strong MARKOV property. It is also pathwise unique and strong, up until the first time

$$\mathcal{S} := \inf\{t > 0 : X_1(t) = X_2(t) = X_3(t)\}$$

a triple collision occurs; however, both pathwise uniqueness and strength fail after time $\mathcal{S}$.

- *"The two Brownian motions can eventually squeeze the ballistic motion in the middle"*, and thus triple points can occur; in fact, *with probability one in the case* $\delta_1 = \delta_2 = \delta_3$.
- Proof employs the excursion argument similar to [IKPY] for the Walsh Brownian motion. The sample paths are given in Fig 2.

3. Middle diffusion, ballistic hedges, skew-elastic collisions

We consider with $\delta_1,\ \delta_2,\ \delta_3,\ x_1 > x_2 > x_3$ given real numbers, the system of equations, first introduced and studied in [F] : for $i = 1, 2, 3$.

$$X_i(\cdot) \,=\, x_i + \sum_{k=1}^{3} \delta_k \int_0^{\cdot} \mathbf{1}_{\{X_i(t) = R_k(t)\}}\, \mathrm{d}t + \int_0^{\cdot} \mathbf{1}_{\{X_i(t) = R_2(t)\}}\, \mathrm{d}B_i(t)$$

$$+ \int_0^{\cdot} \mathbf{1}_{\{X_i(t) = R_2(t)\}}\, \mathrm{d}L^{R_2 - R_3}(t) + \int_0^{\cdot} \mathbf{1}_{\{X_i(t) = R_3(t)\}}\, \mathrm{d}L^{R_2 - R_3}(t)\,.$$

Proposition 4. [Invariance probability measure] Under the conditions

$$3\,\delta_3 \,>\, 2\,\delta_1 + \delta_2\,, \qquad 2\,\delta_3 \,>\, \delta_1 + \delta_2\,,$$

both $\lambda_1 := 2(3\delta_3 - 2\delta_1 - \delta_2)$, $\lambda_2 := 2(2\delta_3 - \delta_1 - \delta_2)$ are positive constants and the unique invariant probability measure $\boldsymbol{\pi}$ for the vector process $\big(G(\cdot) = R_1(\cdot) - R_2(\cdot), H(\cdot) = R_2(\cdot) - R_3(\cdot)\big)$ of gaps is the product of exponentials

$$\boldsymbol{\pi}(\mathrm{d}g, \mathrm{d}h) \,=\, 4\,\lambda_1\,\lambda_2\, e^{-2\,\lambda_1 g - 2\,\lambda_2 h}\, \mathrm{d}g\, \mathrm{d}h\,, \qquad (g,h) \in (0,\infty)^2\,.$$

Part of research is supported by the National Science Foundation under grants NSF-DMS-2008427 and NSF-DMS-1615229.

References

[F] FERNHOLZ, E.R. (2011) Time reversal for an $n = 3$ degenerate system. Technical Report, INTECH Investment Management. Princeton, New Jersey.

[IK] ICHIBA, T., KARATZAS, I. (2021) Degenerate Competing Three-Particle Systems. arXiv: 2006.04970 to appear in *Bernoulli*.

[IKPY] ICHIBA, T., KARATZAS, I., PROKAJ, V., YAN, M.. (2018) Stochastic integral equations for Walsh semimartingales. *Ann. Henri Poincaré* (B) **54** 726–756.