Topics in Multi-dimensional Diffusion Theory: Attainability, Reflection, Ergodicity and Rankings

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ABSTRACT

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This thesis is a contribution to multi-dimensional diffusion theory. Attainability, ergodicity, and rankings of n-dimensional diffusions will be discussed in the intersection of the theories of elliptic partial differential equations and of the stochastic calculus. The idea of effective dimension for diffusions, originally explored in the theory of the exterior Dirichlet problem, gives a criterion for the attainability of an (n-2)-dimensional hyperplane.

This attainability can be rephrased as a triple-collision problem of n diffusive interacting particles on the real line. Another criterion for the attainability comes from the so-called skew-symmetry condition of Brownian motion with oblique reflection in the (n-1)-dimensional positive orthant. Non-attainability plays a crucial role not only in uniqueness of the diffusion in the sense of probability distribution but also in determining related one-dimensional local times of continuous semi-martingales.

These considerations have ramifications concerning the ergodic properties of ranked diffusion obtained from those of the (n-1)-dimensional Brownian motion with reflection. These topics will be united in a fresh manner with an application to the mathematical study of the Atlas model of equity market.

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Chapter 1

SDEs with Bounded Coëfficients

It is very difficult to capture *all* random phenomena in the world. Unfortunately for politicians or rulers who must face random outcomes and their consequences, and fortunately for researchers who are fascinated by random events and their mathematical structure, observed randomness is too complicated to do so. Theories of Probability and Statistics have provided useful frameworks and tools to understand randomness by (i) simplifying the way of thinking towards randomness, (ii) setting up a mathematical model which describes the randomness, (iii) showing consequences in the model and (iv) seeking a closer model to the reality. There are many techniques of simplification. One of them is to make variables constants. It often works very well, while it is sometimes too simple to be used in practice. Other useful simplifications are, for examples,

- removal or replacement of variables by new objects
- approximation by linear/smooth/convex/bounded objects

and so on. Each describes different aspects and levels of randomness and has advantages and disadvantages because of complication of random phenomena.

In this dissertation we focus on the last one of simplifications in the study of stochastic dynamic systems, namely, a system of stochastic differential equations (SDEs) with *bounded* coëfficients. This simplification is still complicated much more than expected and the resulting model has mathematically rich enough structure. Our goal is to determine how far the model can depart from the constant-coëfficient model and how closer it can be to the reality.

The system is well-defined and uniquely determined in the sense of probability distribution under some conditions. We organize important results derived by researchers previously and then state some new observations and contributions to the study of the system.

We introduce some notations and motivating examples in Section 1.1. Then we shall study existence and uniqueness of process with bounded measurable coëfficients. In Section 1.2 the existence and uniqueness are discussed through the martingale problem studied by Stroock and Varadhan [55].

The existence of solution is shown by approximating the target probability measure by a sequence of probability measures. The weak convergence of the sequence of probability measures is verified from some estimates. The estimates are on the expectation of the integral of the bounded function of coördinate process with respect to time under the approximating sequence of probability measures. A key estimate is given in Proposition 1.2 obtained by Aleksandrov [4]. Here, we recite it with a shortcut, preserving Aleksandrov's beautiful geometric approach in Section 1.3.

When we discuss uniqueness of the process in the weak sense, we encounter difficulties of dealing the process which enters regions of discontinuity of coëfficients. Section 1.4 is devoted to *attainability* of sets for the process studied mainly by Friedman [13] with a counter-intuitive example given by Bass & Pardoux [9]. The study of uniqueness for *piecewise constant coëfficients* in polyhedral domains by Bass & Pardoux [9] is explained with Kreĭn-Rutman theory [33] of convex cones in Section 1.5.

After these verifications of well-posedness of the process, the *triple-collision* problem of process is introduced and partially answered in Section 1.6. We introduce the idea of *effective dimensions* (1.127) of multidimensional diffusions, which originally comes form the study of exterior Dirichlet problem explored by Meyer and Serrin [44]. This part of Section 1.6 is one of new contributions to the current subject by the current author. The understanding of triple-collision problem leads us to the consideration of local times described in the next chapters. In Chapter 3 we come back to the same triple collision problem but with another view obtained from the study of Brownian motion with reflection in the positive orthant discussed in Chapter 2.

1.1 Introduction

In this section we introduce some notations and definitions used throughout this dissertation. Some motivating examples of stochastic dynamical systems in the class of stochastic differential equations with bounded coëfficients are listed for later references.

1.1.1 Basic Notations

Probability space

Let $n \geq 2$ be an integer, Ω be the set $C(\mathbb{R}_+;\mathbb{R}^n)$ of all continuous functions from the positive real line $\mathbb{R}_+$ to $n-$dimensional Euclidian space $\mathbb{R}^n$. For each $\omega \in \Omega$, the function $\omega : t \in \mathbb{R}_+ \mapsto \omega(t) \in \mathbb{R}^n$ is the restriction of the function at the "time" t. Let $\mathcal{F}$ be the $\sigma-$field generated by Ω. Define σ-fields $\mathcal{F}_t = \sigma(\omega(s), 0 \leq s \leq t)$ associated with $\omega \in \Omega$ for $t \in \mathbb{R}_+$, and define the filtration $\mathbb{F} = (\mathcal{F}_t)_{t\in\mathbb{R}_+}$. Let $\mathbb{P}_x$ is the Wiener measure on the filtered space $(\Omega, \mathbb{F}, \mathcal{F})$ such that $\mathbb{P}_x(\omega(0) = x) = 1$. Now consider the class $\mathcal{M}$ of all probability measures μ on $\mathbb{R}^n$ and define the initial measure $\mathbb{P}_\mu(A) = \int_{\mathbb{R}^n} \mathbb{P}_x(A)\, d\,\mu(x)$ for $A \in \mathcal{F}$. We re-write $\mathbb{F}$ for the completion with respect to $\cup_{\mu\in\mathcal{M}}\mathbb{P}_\mu$.

Matrices

All the vectors are defined as the column vectors unless it is to be specified. The superscript $\cdot'$ for matrices indicates their transposition. Let $\mathbf{1}$ be the $(n \times 1)$ vector of ones.

Differentials

Let us denote the class of continuously differentiable functions having all derivatives of order smaller than or equal to k in $\mathbb{R}^n$ by $C^k(\mathbb{R}^n)$. Let $C_0^k(\mathbb{R}^n)$ be the class of functions in $C^k(\mathbb{R}^n)$ with compact support in $\mathbb{R}^n$. We denote the gradient, Hessian and Laplacian operators by ∇, H and Δ respectively, i.e., for any function $\phi \in C^2(\mathbb{R})$

$$\nabla\phi(x) := \frac{\partial\phi}{\partial x}(x) = \Big(\frac{\partial\phi}{\partial x_1}, \ldots, \frac{\partial\phi}{\partial x_n}\Big)', \quad H\phi(x) := \big(D^2_{ij}\phi(x)\big)_{\{1\leq i,j\leq n\}} = \Big(\frac{\partial^2\phi}{\partial x_i \partial x_j}\Big)_{\{1\leq i,j\leq n\}},$$
$$\Delta\phi(x) := \mathbf{1}'\,(H\phi)(x)\,\mathbf{1} = \sum_{i=1}^n\sum_{j=1}^n \frac{\partial^2\phi}{\partial x_i\partial x_j}.$$

Wedges in $\mathbb{R}^n$

We shall construct a map $\mathcal{P} : \mathbb{R}^n \mapsto \mathbb{R}_+^{n-1} := \{\, y \in \mathbb{R}^{n-1};\, y_i \geq 0,\ i = 1, \ldots, n-1\}$ in the next three paragraphs. Hereafter we use $\mathbb{R}_+^{n-1}$ in this sense which is different from the common notation.

Let Π be the symmetric group of permutations of $\{1, \ldots, n\}$. For example, two elements $\pi, \pi' \in \Pi$ with $\pi(1) = 1, \pi(2) = 2, \pi(3) = 3, \ldots, \pi(n) = n$ and $\pi'(1) = 2, \pi'(2) = 1, \pi'(3) = 3, \ldots, \pi'(n) = n$ are different. The set Π consists of $n!$ elements.

Let us define a map $\mathfrak{p}^x : \mathbb{R}^n \mapsto \Pi$ such that for each $x \in \mathbb{R}^n$

$$x_{\mathfrak{p}^x(1)} \geq x_{\mathfrak{p}^x(2)} \geq \cdots \geq x_{\mathfrak{p}^x(n)}\,, \tag{1.1}$$

where the ties are resolved by choosing the smaller index for the bigger i.e. $\mathfrak{p}^x(j) < \mathfrak{p}^x(j+1)$ if $x_{\mathfrak{p}^x(j)} = x_{\mathfrak{p}^x(j+1)}$ for some $j = 1, \ldots, n-1$. For every $\pi \in \Pi$ we can define a polyhedral region $\mathcal{R}_\pi$ by

$$\mathcal{R}_\pi := \{x \in \mathbb{R}^n \, ; \, x_{\pi(1)} \geq x_{\pi(2)} \geq \cdots \geq x_{\pi(n)}\}\,. \tag{1.2}$$

Note that the union of all such $\mathcal{R}_\pi$ over $\pi \in \Pi$ is the whole space, i.e., $\cup_{\pi\in\Pi}\mathcal{R}_\pi = \mathbb{R}^n$. The interiors $\overset{\circ}{\mathcal{R}}_\pi$ and $\overset{\circ}{\mathcal{R}}_{\pi'}$ of different regions are disjoint, i.e., $\overset{\circ}{\mathcal{R}}_\pi \cap \overset{\circ}{\mathcal{R}}_{\pi'} = \emptyset$ for $\pi \neq \pi'$, $\pi, \pi' \in \Pi$. We define $Q_i^{(k)} := \cup_{\pi:\pi(k)=i}\mathcal{R}_\pi$ where the ith component is ranked kth for $i, k = 1, \ldots, n$.

A projection from $\mathbb{R}^n$ to $\mathbb{R}^{n-1}$

Now let us define a projection map $\mathcal{P} : \mathbb{R}^n \mapsto \mathbb{R}_+^{n-1}$ by

$$\mathcal{P}x := (x_{\mathfrak{p}^x(1)} - x_{\mathfrak{p}^x(2)}, x_{\mathfrak{p}^x(2)} - x_{\mathfrak{p}^x(3)}, \ldots, x_{\mathfrak{p}^x(n-1)} - x_{\mathfrak{p}^x(n)}) \in \mathbb{R}_+^{n-1}\,; \quad x \in \mathbb{R}^n\,. \tag{1.3}$$

1.1.2 Bounded Continuous Coëfficients

Let us define an $n-$dimensional Ito process by $X := (\,(X_1(t), \ldots, X_n(t)\,)', t \in \mathbb{R}_+)$ which is given by the stochastic differential equations in a schematic matrix form:

$$d\,X(t) = b(X(t))\,d\,t + s(X(t))\,d\,W(t)\,; \quad 0 \leq t < \infty\,, \qquad X(0) = x_0 \in \mathbb{R}^n \tag{1.4}$$

where the measurable functions $b(\cdot) : \mathbb{R}^n \to \mathbb{R}^n$ and $s(\cdot) : \mathbb{R}^{n\times n}$ are bounded and continuous, and $W = ((W_1(t), \ldots, W_n(t))', t \in \mathbb{R}_+)$ is the n-dimensional standard Brownian motion on some filtered probability space $(\Omega\,, \mathcal{F}\,, \mathbb{F}\,, \mathbb{P})$. It is well known that if the coëfficients satisfy the global Lipschitz and linear growth conditions

$$\begin{aligned} &\|b(x) - b(y)\| + \|s(x) - s(y)\| \leq K\|x - y\|\,, \\ &\|b(x)\|^2 + \|s(x)\|^2 \leq K^2(1 + \|x\|^2)\,, \end{aligned} \tag{1.5}$$

for every $x, y \in \mathbb{R}^n$, where K is a positive constant, then there exists a continuous, adapted process X exists and square-integrable in the sense that there exists a constant C such that

$$\mathbb{E}\|X(t)\|^2 \le C(1 + \|x_0\|^2)\exp(C\,t)\,; \quad 0 \le t \le T\,, \tag{1.6}$$

for every $T \ge 0$.

Here are some examples.

Example 1.1 (Constant coëfficients)**.** $b(\cdot) = \bar{b}$, $s(\cdot) = \bar{s}$ for some constant vector $\bar{b} \in \mathbb{R}^n$ and matrix $\bar{s} \in \mathbb{R}^{n\times n}$. Although this is a very special case, we can extend our consideration to a more general set-up of process $Y := \{(Y_1(t), \ldots, Y_n(t))\,;\, t \in \mathbb{R}_+\}$ by considering transformations $X = F(Y)$. When $F(\cdot) = \log(\cdot)$, the solution Y becomes the geometric Brownian motion.

Example 1.2 (A class of bounded solutions)**.** Does the boundedness of coëfficient imply boundedness of solution in some sense? Generally not, however, here is a class of bounded solution with bounded coëfficients. For notational simplicity, let $n = 1$. If the functions satisfy $b = (\nabla F)\circ F^{-1}$ and $\sigma = (\Delta F)\circ F^{-1}$ for some invertible function $F : \mathbb{R} \mapsto \mathbb{R}$ of a class C_b^2 of twice continuously differentiable bounded functions, then the solution is $X(t) = F(W(t))$ for $0 \le t < \infty$. Especially, the solution is bounded. For example, if $F(x) = \Phi(x) := \int_{-\infty}^{x} \phi(y)\,dy$ where ϕ is Gaussian kernel $\phi(x) = (2\pi)^{-1}\exp(-x^2/2)$, $x \in \mathbb{R}$, then $X(\cdot) = \Phi(B(\cdot))$ is the solution.

1.1.3 Piecewise Constant Coëfficients

Now suppose that the measurable functions $b(\cdot) : \mathbb{R}^n \mapsto \mathbb{R}^n$ and $s(\cdot) : \mathbb{R}^{n\times n}$ in (1.4) are piecewise constant in each polyhedral region $\mathcal{R}_\pi, \pi \in \Pi$ of (1.2) with the map $\mathfrak{p}^x$ in (1.1), i.e.,

$$b(x) = \sum_{\pi\in\Pi} b_\pi 1_{\mathcal{R}_\pi}(x) = b_{\mathfrak{p}^x}\,, \quad s(x) = \sum_{\pi\in\Pi} s_\pi 1_{\mathcal{R}_\pi}(x) = s_{\mathfrak{p}^x}\,; \quad x \in \mathbb{R}^n\,, \tag{1.7}$$

with constant vectors b_π and some nonsingular constant matrix s_π for every $\pi \in \Pi$. The matrix-valued functions $b(\cdot)$ and $s(\cdot)$ are bounded and measurable but not necessarily continuous. In each region $\mathcal{R}_\pi$ the process X behaves like a diffusion of constant coefficient. Here are examples of such system.

Example 1.3 (Monotone drift and Atlas model)**.** Suppose that for each $\pi \in \Pi$, the drift coefficient b_π in (1.4), (1.7) satisfies

$$b_{\pi(1)} \le b_{\pi(2)} \le \ldots \le b_{\pi(n)} . \tag{1.8}$$

Then, for each $t \ge 0$, $1 \le i < j \le n$ the smaller $X_{\pi(j)}(t)$ has larger drift $b_{\pi(j)}$ than the larger $X_{\pi(i)}(t)$. Moreover, if all $b_{\pi(j)}$ except for $j = n$ are negative and $b_{\pi(n)}$ is positive with

$$b_{\pi(1)} + \cdots + b_{\pi(n)} = 0 ; \quad \pi \in \Pi , \tag{1.9}$$

then the model becomes one special case of financial equity markets studied as the so-called Atlas model by Banner, Fernholz & Karatzas [5]. □

Example 1.4 (Concave variance and Linearly growing variance)**.** Suppose that for each $\pi \in \Pi$, the volatility coefficient s_π in (1.4), (1.7) satisfies

$$s_\pi = \mathrm{diag}(\sigma_{\pi(1)}, \ldots, \sigma_{\pi(n)}), \quad \sigma^2_{\pi(2)} - \sigma^2_{\pi(1)} \ge \sigma^2_{\pi(3)} - \sigma^2_{\pi(2)} \ge \cdots \ge \sigma^2_{\pi(n)} - \sigma^2_{\pi(n-1)} . \tag{1.10}$$

Then we say that the volatility structure has concave variance in the sense that, for example, the smaller $X_{\pi(i)}(t)$ has more variance $\sigma^2_{\pi(i)}$ than the larger $X_{\pi(j)}(t)$ for each $t \ge 0$, $1 \le i < j \le n$ but the increments become smaller and smaller, as the ranks go further down. Moreover, if the inequalities hold as equalities, that is for some constants $\sigma_0 > 0$ and $\mathfrak{s} \in \mathbb{R}$

$$0 < \sigma^2_{\pi(k)} = \sigma_0 + k\,\mathfrak{s} ; \quad k = 1, \ldots, n . \tag{1.11}$$

we call the model has linearly growing variance structure. □

Those examples are very special yet have some important parametric structures which we look into deeper both with some theoretical interest in stochastic processes and with its application to mathematical finance. The condition (1.9) gives us the insight that the system does not move far away quickly, when the time t goes to infinity. With this condition and other conditions will give us some ergodic properties of the system. The linearly growing variance condition was observed by Fernholz [12] in the actual U.S. equity market. Later we study the so-called no-triple-collision problem where the volatility structure is of great importance. The linearly growing condition (1.10) is a sufficient condition for absence of triple collisions.

1.1.4 Piecewise Continuous Coëfficients

We may work a little bit more general than piecewise constant case. Here is an example where its uniqueness is known.

Example 1.5 (Gao [14])**.** Suppose that the coëfficient $s(\cdot)$ in (1.4) is piecewise continuous on each of the two half spaces, i.e., if the $(i,j)-$th element satisfies

$$s_{ij}(\cdot) = \begin{cases} s_{ij}^{+}(\cdot) & \text{in } \mathbb{R}_{+}^{n} = \{x \in \mathbb{R}^n \,:\, x_1 \geq 0\} \\ s_{ij}^{-}(\cdot) & \text{in } \mathbb{R}_{-}^{n} = \{x \in \mathbb{R}^n \,:\, x_1 < 0\} \end{cases}$$

where $s_{ij}^{\pm} : \mathbb{R}^n \mapsto \mathbb{R}$ are continuous functions. Gao [14] studied this example and showed uniqueness of weak solution. $\square$

1.2 Existence and Uniquness of Martingale Problem

We defined the stochastic model (1.4) as the system of $n-$dimensional SDE. The appropriate next step is to ask whether the system is well defined. Note that the coefficients $b(\cdot)$ and $s(\cdot)$ in the system (1.4) do not satisfy the continuity in general. For example, the Atlas model and concave variance structure in Example 1.3 and 1.4 have discontinuities. This consideration invites us to take a careful examination of the well-posedness of the solution to the stochastic system. If the coefficients are fixed (Example 1.1) or Lipschitz continuous (1.5), we may approximate the system more easily by a sequence of recursive Picard-Lindelof type iterations, as in the classical theory of differential equations. Otherwise, we consult with the partial differential equations theory, in order to extend our considerations to a wider class of solutions in the previous Sections 1.1.3 and 1.1.4.

Existence of weak solutions to the system of SDEs with bounded coëfficients are well-studied by Krylov [35], Stroock & Varadhan [55]. The so-called Alexandroff's estimate for the elliptic partial differential equations plays the essential role in the proof of the existence. The estimates are in terms of L^p-norm for $p \geq n$, since the coëfficients are not necessarily continuous. The Itô's formula and the formulation of martingale problem build a bridge between the system of SDEs and the elliptic partial differential equations with the infinitesimal generator of the Markov process. We combine these techniques to show existence.

Uniqueness has another story. Here we consider uniqueness in the weaker sense of probability distribution, since there strong uniqueness does not hold in general. The difficulty in showing uniqueness lies on the behavior around the vertex and the boundary of the regions where the coëfficients have discontinuities. The process may visit a point, for example, the origin in the $n-$dimensional coordinates infinitely often with probability one; see Section 1.4.

Let us explain this some more. The $n-$dimensional standard Brownian motion visits the origin if $n = 1$ infinitely often, while it never visits the origin if $n \geq 2$, with probability one. It is shown by Girsanov's change of measure argument that for the $n-$dimensional Brownian motion with constant drift and variance-covariance rate shares the same property. However, as in the example of Bass & Pardoux [9], it is not true for the $n-$dimensional diffusion in general. In fact, Bass & Paroux constructed an example such that the $n-$dimensional diffusion X with piecewise constant coefficients in each conic region but not exactly same as (1.4)-(1.7) hits the origin infinitely often with probability one. They chose the region and the variance-covariance rate in a specific way so that each region is a polyhedral cone with the vertex being the origin with carefully chosen small aperture and all the eigenvalues of the variance-covariance matrix is small except one direction. Their construction is explained with some modification in Section 1.4.1. If the process is attracted to the vertex, the understanding around the vertex needs more effort.

Therefore, the proof of uniqueness of the weak solution requires more delicate arguments near the vertex. Bass & Pardoux [9] have overcome this difficulty up to some extent. They used Kreĭn-Rutman theorem for the positive compact operator in a cone to compute the resolvent for the distribution of the diffusion. It is shown that the resolvent is the limit of ratios of integrals with respect to Markov transition probabilities and is *uniquely* determined by the drift $b(\cdot)$ and diffusion $s(\cdot)$ coëfficients. The uniqueness of the process follows from this uniqueness.

In the next few sections we discuss the following Theorem 1.1.

Theorem 1.1 (Existence [35], [55] ; Uniqueness [9])**.** *The weak solution to* (1.4)-(1.7) *exists and is unique in the sense of probability distribution.*

Denote the distribution of the solution starting at $x_0 \in \mathbb{R}^n$ by $\mathbb{P}_{x_0}$.

1.2.1 Martingale Problem

In this section we discuss how the existence and uniqueness of the system of SDEs defined by (1.4) and (1.7) are to be understood in the weak sense. The key idea is to transform the problem into the so-called martingale problem initiated by Stroock & Varadhan[55].

Let $L_{a,b}$ be the second-order differential operator defined on the space of twice continuous function $\phi \in C^2(\mathbb{R}^n)$ by

$$L_{a,b}\,[\phi](x) = \frac{1}{2}\sum_{i,j=1}^{n} a_{ij}(x)\frac{\partial^2 \phi}{\partial x_i \partial x_j}(x) + \sum_{i=1}^{n} b_i(x)\frac{\partial \phi}{\partial x_i}(x)\,; \quad x \in \mathbb{R}^n\,, \tag{1.12}$$

where $a_{ij}(\cdot)$ and $b_i(\cdot)$ are bounded measurable functions, and assume that the matrix-valued function $a(\cdot) = (a_{ij}(\cdot))$ is uniformly elliptic in $\mathbb{R}^n$. If it is so, the operator is often called strictly elliptic or uniformly elliptic.

The elliptic operator L can be seen as the infinitesimal generator of Markov process X. The corresponding process can be written as in the stochastic differential form:

$$d\,X(t) = b(X(t))\,d\,t + \sigma(X(t))\,d\,W(t)\,; \quad 0 \le t < \infty\,, \tag{1.13}$$

where $\sigma(\cdot)$ is the square-root of the $(n \times n)$-matrix-valued function $a(\cdot)$, i.e., $a(\cdot) = \sigma(\cdot)\sigma(\cdot)'$. We have in mind the special case when $a(\cdot) \equiv s_{\mathfrak{p}^x}\, s'_{\mathfrak{p}^x}$ and $b(x) \equiv b_{\mathfrak{p}^x}$ which is piecewise constant in each polyhedral domain defined in (1.1) and (1.4).

We review with the definition of weak solution.

Definition 1.1 (Weak Solution)**.** A weak solution of equations is a triplet (X, W), $(\Omega, \mathcal{F}, \mathbb{P})$, $\mathbb{F} = \{\mathcal{F}_t\}$ where

(i) $(\Omega, \mathcal{F}, \mathbb{P})$ is a probability space, and $\mathbb{F}$ is a filtration of sub-σ-fields of $\mathcal{F}$ satisfying the usual conditions

(ii) $X = \{X_t, \mathcal{F}_t; 0 \le t < \infty\}$ is a continuous, adapted $\mathbb{R}^n$-valued process and $W = \{X_t, \mathcal{F}_t; 0 \le t < \infty\}$ is an n-dimensional Brownian motion,

(iii) for $1 \le i, j \le n$

$$\mathbb{P}\left[\int_0^t (\,|b_i(X(u))| + |\sigma_{ij}(X(u))|^2\,)d\,u < \infty\right] = 1 \qquad \text{and} \tag{1.14}$$

(iv)

$$X(t) = X(0) + \int_0^t b(X(u))\,d\,u + \int_0^t \sigma(X(u))\,d\,W(u)\,, \quad 0 \le t < \infty\,, \mathbb{P} - a.s. \tag{1.15}$$

To find a weak solution is to find such a Brownian motion in the definition. P. Lévy's characterization of Brownian martingales says that any vector of $n-$dimensional square-integrable continuous martingales with all quadratic variation processes linear in time and with zero cross-variation processes is $n-$dimensional Brownian motion [42]. In fact, we have the following Lemma 1.1.

Lemma 1.1 (Problem 4.4 of Karatzas & Shreve [28]). *Let* $X := \{(X(t), \mathcal{F}_t) ; \quad 0 \le t < \infty \}$ *be a continuous adapted process. For any function* $f(\cdot)$ *in the space* $C^2(\mathbb{R})$ *of twice continuously differentiable functions on* $\mathbb{R}$ *,*

$$f(X(t)) - f(X(0)) - \int_0^t f''(X(s))\, d\, s\,, \quad \mathcal{F}_t\,;\, 0 \le t < \infty\,, \tag{1.16}$$

is a continuous local martingale, if and only if the process $X(\cdot)$ *is a Brownian motion.*

Thus, we consider a class of square-integrable martingales first, instead of Brownian motion itself, in the following Problem 1.

Problem 1 ((Local) Martingale Problem). *The (local) martingale problem is to find a probability measure* $\mathbb{P}$ *on some probability space* $(\Omega = C(\mathbb{R}_+, \mathbb{R}^n), \mathcal{F} := \mathcal{B}(\Omega), \mathbb{F})$ *such that* $\mathbb{P}_{x_0}(X(0) = x_0) = 1$ *, and*

$$f(X(t)) - f(X(0)) - \int_0^t L_{a,b} f(X(s))\, d\, s\,; \quad 0 \le t < \infty$$

is a $\mathbb{P}_{x_0}$*-(local) martingale for all* $f \in C^2(\mathbb{R}^n)$ *and all* $x_0 \in \mathbb{R}^n$*.*

If the local martingale problem is solved, we use the following Proposition 1.1. Note that if the volatility coefficients $\sigma(\cdot)$ is bounded, the local martingale problem can be shown to be equivalent to the martingale problem.

Proposition 1.1 (Local Martingale Problem and Weak Solution. Theorem 4.2.1 of [55]; Proposition 4.6 of [28]). *Suppose that a probability measure* $\mathbb{P}$ *is a solution to the local martingale problem associated with* $L_{a,b}$ *in (1.12). Then, there is an $n-$dimensional Brownian motion* $W := (W(t), \mathcal{F}(t); 0 \le t < \infty)$ *defined on an extension* $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{\mathbb{P}})$ *of* $(\Omega, \mathcal{F}, \mathbb{P})$ *such that* (X, W) *,* $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{\mathbb{P}})$ *,* $\tilde{\mathbb{F}} := \{\tilde{\mathcal{F}}_t\}$ *is a weak solution.*

1.2.2 Bounded Continuous Coëfficients

When the coëfficients are bounded and continuous, we have the fundamental existence and uniqueness result for weak solutions of SDE (1.13).

Theorem 1.2. [Skorohod [51], [53]; Stroock & Varadhan [55]] *There exists a weak solution of* (1.13) *when the coëfficients* $b(\cdot)$ *and* $\sigma(\cdot)$ *are bounded and continuous, and* $\sigma(\cdot)$ *is non-negative definite. Moreover, if the Cauchy problem*

$$\frac{\partial u}{\partial t} = L_{a,b} u \,; \qquad in\ (0, \infty) \times \mathbb{R}^n \,, \quad u(0, \cdot) = f \,; \qquad in\ \mathbb{R}^n \tag{1.17}$$

has a solution $u^f \in C([0, \infty) \times \mathbb{R}^n) \cap C^{1,2}((0, \infty) \times \mathbb{R}^n)$ *which is bounded on each strip of the form* $[0, T] \times \mathbb{R}^n$ *, for every* $f \in C_0^\infty(\mathbb{R}^n)$ *, then there exists at most one solution of* (1.13).

The proof of existence uses approximation of solution $\mathbb{P}$ by a tight sequence $\{\mathbb{P}^{(n)}\}$ of solutions for SDE with step-function like coëfficients $b^{(n)}$ and $\sigma^{(n)}(\cdot)$, which approximate the bounded continuous $b(\cdot)$ and $\sigma(\cdot)$. We use both boundedness and continuity for the tightness of $\{\mathbb{P}^{(n)}\}$.

It is interesting to see the duality between existence of the solution to the Cauchy problem (1.17), and uniqueness of the solution to SDE (1.13). A sufficient condition for existence of bounded solution u^f on each strip of the form $[0, T]$ to the Cauchy problem (1.17), is that the coëfficients be bounded and Hölder-continuous on $\mathbb{R}^n$ and the matrix value function $a(\cdot)$ in (1.12) be uniformly positive definite.

At this point one can naturally pose the following problem.

Problem 2. *Can we relax the conditions of continuity of coëfficients for existence and uniqueness in the above Theorem 1.2?*

1.2.3 Existence

In the following, let us first examine the existence based on a thorough study by N.V. Krylov in a sequence of papers [38], [39], [40], [34] [36]. Also, we refer Exercise 7.3.2 of Stroock & Varadhan [55].

This section sketches the proof of the existence of weak solutions. The details are explained in the following subsections. First, we remove the drift part. This is done through Girsanov's change-of-measure theorem because the drift coëfficients are bounded and measurable.

Removal of Drift

Let us define the process

$$\xi(t) := \sigma^{-1}(X(t))\, b(X(t)) \,, \qquad 0 \le t < \infty \,.$$

Assume that the process $\xi(\cdot)$ is progressively measurable. In fact, when the measurable functions $b(\cdot)$ and $s(\cdot)$ in (1.4) are piecewise continuous, by the nature of the functions $b(\cdot)$ and $\sigma(\cdot)$, the mapping $t \mapsto \xi(t)$ is bounded and right-continuous or left-continuous on each boundary $\partial\mathcal{R}_{\mathfrak{p}(X(t))}$ at time t, deterministically, according to the position $\mathcal{R}_{\mathfrak{p}(X(t-))}$ of $X(t-)$. Then, although the sample path of $n-$dimensional process $\xi(\cdot)$ is not entirely right-continuous or left-continuous, it *is* progressively measurable. Moreover, $\xi(\cdot)$ is bounded, so the exponential process

$$\eta(t) = \exp\left[\sum_{i=1}^{n}\int_0^t \langle \xi_i(u), d\,W_i(u)\rangle - \frac{1}{2}\int_0^t \|\xi(s)\|^2 d\,u\right]; \quad 0 \le t < \infty$$

is a continuous martingale, where $\|x\|^2 := \sum_{j=1}^n x_j^2$, $x \in \mathbb{R}^n$ stands for $n-$dimensional Euclidean norm and the bracket $\langle x, y\rangle := \sum_{j=1}^n x_j y_j$ is the inner product of two vectors $x, y \in \mathbb{R}^n$. By Girsanov's theorem

$$\widetilde{W}(t) := W(t) + \int_0^t \sigma^{-1}(X(u))\mu(X(u))d\,u\,,\, \mathcal{F}_t\,; \quad 0 \le t < \infty \tag{1.18}$$

is an n-dimensional standard Brownian motion under the new probability measure $\mathbb{Q}$ that satisfies by $\mathbb{Q}(C) = \mathbb{E}^{\mathbb{P}}\big(\eta(T)1_C\big)$ for $C \in \mathcal{F}_T^W$, $0 \le T < \infty$.

Thus, it suffices to consider the case of $b(\cdot) \equiv 0$, namely,

$$X(t) \;=\; x_0 + \int_0^t \sigma\big(X(u)\big)\,dW(u)\,, \qquad 0 \le t < \infty\,. \tag{1.19}$$

By this removal of the drift part, it is essential to handle the case when $b(\cdot) \equiv 0$, i.e., the second-order differential operator for $\phi \in C^2(\mathbb{R}^n)$ is

$$L\,[\phi](x) = \frac{1}{2}\sum_{i,j=1}^{n} a_{ij}(x)\frac{\partial^2\phi}{\partial x_i \partial x_j}(x)\,; \quad x \in \mathbb{R}^n\,. \tag{1.20}$$

Application of Alexandrov's Estimates

The idea of showing existence is to find a probability measure which satisfies Definition 1.1. This is done through an approximation procedure of the target probability measure $\mathbb{P}$ by a sequence of probability measures $\{P^{(k)}\,,\, k = 1, \ldots\}$ which are solutions to martingale problem for bounded continuous function $\tilde{a}^{(k)} : \mathbb{R}^n \to S^n$, $k = 1, \ldots$ In order to do so, we apply a priori estimate for partial differential equations obtained first by A.D. Aleksandrov [4] and obtain the following Proposition 1.2.

Proposition 1.2 ([36]; Exercise 7.3.2 of [55]). *Suppose that* $\widetilde{a}^{(k)} : \mathbb{R}^n \to \mathcal{S}^n$ *is a bounded continuous matrix-valued function and uniformly elliptic, i.e., there exist some constants* $\Lambda \geq \lambda > 0$ *such that*

$$\Lambda|\theta|^2 \geq \theta' \widetilde{a}^{(k)}(x)\theta \geq \lambda|\theta|^2 ; \quad \theta \in \mathbb{R}^n , \, x \in \mathbb{R}^n .$$

Then the solution to martingale problem for

$$\widetilde{L}^{(k)} := \frac{1}{2} \sum_{i,j=1}^{n} \widetilde{a}^{(k)}_{ij}(x) \partial^2 / \partial x_i \partial x_j \tag{1.21}$$

starting from $x \in \mathbb{R}^n$ *exists. Moreover, for all* $p (> n)$, $T > 0$, $R > 0$, $f \in C_0(\mathbb{R}^n)$ *with support* $supp(f) \subset B_0(R)$ *in the ball with center* 0 *and radius* R,

$$\left| \mathbb{E}^{\mathbb{P}^{(k)}} \Big[\int_0^T f(X(t)) \, dt \Big] \right| \leq C_{\lambda,\Lambda,p,T} \|f\|_{L^p(\mathbb{R}^n)} , \tag{1.22}$$

where $C_{\lambda,\Lambda,p,T}$ *is a constant which depends on* λ, Λ, p, T *only.*

The estimate (1.22) can be derived mainly based on the consequence of Monge-Ampère equation and λ-concave functions studied by Aleksandrov and Krylov. We recite one of their representative estimates as (1.54) in Proposition 1.2 of Section 1.3.3 where we present it in a shortcut manner, since the original proof was lengthy and scattered. Aleksandrov [1] [2] [3] gave a geometric approach to the Dirichlet problem of second-order partial differential equations:

$$Lu = f \quad \text{in } G , \qquad u = 0 \quad \text{on } \partial G \tag{1.23}$$

for a bounded subset $G \subset \mathbb{R}^n$ and the second-order differential operator L defined in (1.20). With Aleksandrov's clever observation on convexity and projection the solution to the Dirichlet problem is bounded by L^p-norm for $p \geq n$. In fact, Aleksandrov obtained that under some conditions

$$|u(\cdot)| \leq C \|f\|_{L^p} ; \quad \text{in } \bar{G} \tag{1.24}$$

for some constant C depending on the minimum and maximum of eigenvalues of matrix-valued function $a(\cdot)$ and the bounded region G. Following the idea of Aleksandrov, Krylov examined the results extensively and analytically in view of the maximum principle of partial differential equation and the stochastic control problem.

Sketch of Proof of Existence of Solution to (1.4) *with bounded measurable coëfficients.* Here we take Proposition 1.2 for granted temporarily. We choose the coefficients $\{\tilde{a}^{(k)}(\cdot)\}$ to be a sequence of bounded continuous maps which satisfy that for any fixed $q \geq 1$, $R > 0$ and $\varepsilon > 0$, there exists k_0 such that

$$\Big| \int_{B_0(R)} ||\tilde{a}^{(k)}(x) - a(x)||^q \, d\,x \Big| < \varepsilon \, ; \quad k \geq k_0, \tag{1.25}$$

Here $\|\cdot\|$ is the matrix norm. Since the $a^{(k)}$'s are bounded independent of k, the solutions $\{\widetilde{\mathbb{P}}_x^{(k)}\}$ to the martingale problem for $\widetilde{L}^{(k)}$ in (1.21) starting from $x \in \mathbb{R}^n$ form a weakly conditionally compact set in the space of probability measures on $C(\mathbb{R}_+, \mathbb{R}^n)$. Let $\{\widetilde{\mathbb{P}}^{(k')}\}$ be a convergent subsequence of $\{\widetilde{\mathbb{P}}^{(k)}\}$ and let $\mathbb{P}$ be its limit.

For any bounded continuous $\Phi : \mathbb{R} \mapsto \mathbb{R}$, and $f \in C_0^\infty(\mathbb{R}^n)$ with bounded support $\mathrm{supp}(f) \subset B_0(R)$, by the triangle inequality we obtain

$$\begin{aligned}
&\Big|\mathbb{E}^{\widetilde{\mathbb{P}}^{(k)}}\Big[\Phi\Big(\int_0^T \widetilde{L}^{(k)} f(X(t))\, d\,t\Big)\Big] - \mathbb{E}^{\mathbb{P}}\Big[\Phi\Big(\int_0^T L f(X(t))\, d\,t\Big)\Big]\Big| \\
\leq\ & \Big|\mathbb{E}^{\widetilde{\mathbb{P}}^{(k)}}\Big[\Phi\Big(\int_0^T \widetilde{L}^{(k)} f(X(t))\, d\,t\Big) - \Phi\Big(\int_0^T \widetilde{L}^{(k_0)} f(X(t))\, d\,t\Big)\Big]\Big| \\
&+ \Big|\mathbb{E}^{\widetilde{\mathbb{P}}^{(k)}}\Big[\Phi\Big(\int_0^T \widetilde{L}^{(k_0)} f(X(t))\, d\,t\Big)\Big] - \mathbb{E}^{\mathbb{P}}\Big[\Phi\Big(\int_0^T \widetilde{L}^{(k_0)} f(X(t))\, d\,t\Big)\Big]\Big| \\
&+ \Big|\mathbb{E}^{\mathbb{P}}\Big[\Phi\Big(\int_0^T L f(X(t))\, d\,t\Big) - \Phi\Big(\int_0^T L^{(k_0)} f(X(t))\, d\,t\Big)\Big]\Big| .
\end{aligned} \tag{1.26}$$

It follows from (1.22) that the first term of right-hand in (1.26) is estimated by

$$\begin{aligned}
&\Big|\mathbb{E}^{\widetilde{\mathbb{P}}^{(k)}}\Big[\Phi\Big(\int_0^T \widetilde{L}^{(k)} f(X(t))\, d\,t\Big) - \Phi\Big(\int_0^T \widetilde{L}^{(k_0)} f(X(t))\, d\,t\Big)\Big]\Big| \\
&\leq M\ \mathbb{E}^{\widetilde{\mathbb{P}}^{(k)}}\Big[\int_0^T \|\widetilde{a}^{(k)}(X(t)) - \widetilde{a}^{(k_0)}(X(t))\|\, \mathbf{1}_{\{X(t)\in \mathrm{supp}(f)\}}\, d\,t\Big] \\
&\leq M\, C_{\lambda,\Lambda,p,T}\Big[\int_{\mathrm{supp}(f)} \|\widetilde{a}^{(k)}(x) - \widetilde{a}^{(k_0)}(x)\|^p \, d\,x\Big]^{1/p},
\end{aligned} \tag{1.27}$$

where the constant M depends on the bounds on Φ and the second derivatives of $f(\cdot)$. Similarly, the third term of right-hand in (1.26) is estimated by

$$\begin{aligned}
&\Big|\mathbb{E}^{\mathbb{P}}\Big[\Phi\Big(\int_0^T Lf(X(t))\,dt\Big) - \Phi\Big(\int_0^T L^{(k_0)}f(X(t))\,dt\Big)\Big]\Big| \\
&\le M\,C_{\lambda,\Lambda,p,T}\Big[\int_{\mathrm{supp}(f)} \|\widetilde{a}^{(k)}(x) - \widetilde{a}^{(k_0)}(x)\|^p\,dx\Big]^{1/p}.
\end{aligned} \tag{1.28}$$

The second term of right-hand in (1.26) converges to zero, by the continuous mapping theorem:

$$\lim_{k\to\infty}\Big|\mathbb{E}^{\widetilde{\mathbb{P}}^{(k)}}\Big[\Phi\Big(\int_0^T \widetilde{L}^{(k_0)}f(X(t))\,dt\Big)\Big] - \mathbb{E}^{\mathbb{P}}\Big[\Phi\Big(\int_0^T \widetilde{L}^{(k_0)}f(X(t))\,dt\Big)\Big]\Big| = 0\,. \tag{1.29}$$

Thus, combining (1.26), (1.27), (1.28) and (1.29) with (1.25), we obtain

$$\lim_{k\to\infty}\Big|\mathbb{E}^{\widetilde{\mathbb{P}}^{(k)}}\Big[\Phi\Big(\int_0^T \widetilde{L}^{(k)}f(X(t))\,dt\Big)\Big] - \mathbb{E}^{\mathbb{P}}\Big[\Phi\Big(\int_0^T Lf(X(t))\,dt\Big)\Big]\Big| = 0\,. \tag{1.30}$$

Therefore, $\mathbb{P}$ is a solution to the martingale problem for the operator L in (1.20). □

The argument of the above proof is quite general. We can obtain the existence for not only the case of piecewise constant coefficients on the polyhedron domain but also the case of all bounded measurable coefficients. As a summary, we state the following.

Proposition 1.3. *The weak solution to* (1.4) *with bounded measurable coëfficients exists.*

Thus, our task is to derive Proposition 1.2 through the estimates (1.24) for the Dirichlet problem (1.23). There are two closely related approaches to obtain the estimates.

The first one is geometric approach which is the original idea of Aleksandrov. This technique uses the normal image of subset of $\mathbb{R}^n$ with respect to convex function and projection. Since our problem allows the discontinuity of coëfficients, we try to understand what kind of geometric properties are not affected by the discontinuity given in the equation. The idea of Aleksandrov connects the volume of projected convex hulls of solution on different hyper-planes and the solution to the Dirichlet problem.

The second one is the analytic approach studied by Krylov [38] [39] [40] [34] [36] with some relation to stochastic control theory. Both approaches have nice consequences.

In the following sections we state the idea concisely, since the original literature are lengthy and scattered. Then, we prove Proposition 1.2 and complete the proof of Proposition 1.3.

1.3 Geometric Approach to the Dirchlet Problem

In this subsection we review A.D. Aleksandrov's clear geometric idea on the Dirichlet problem (1.23) to derive (1.24) based on [1] [2] [3]. Let us rewrite the Dirichlet problem.

Problem 3. *Find a function $u(\cdot)$ in the space of twice continuously differentiable functions such that for a bounded subset G of $\mathbb{R}^n$, we have*

$$L\,[u](x) = \frac{1}{2}\sum_{i,j=1}^{n} a_{ij}(x)\frac{\partial u}{\partial x_i \partial x_j}(x) = f(x)\,; \quad x \in G \tag{1.31}$$

and $u(x) = 0$ for $x \in \partial G$ where ∂G is the boundary of G.

The classical theory of partial differential equations guarantees the existence of solution to the Dirichlet problem for continuous coëfficient $A(\cdot) = (a_{ij}(\cdot))_{1\le i,j\le n}$ and Hölder continuous data $f(\cdot)$ for a Hölder continuous smooth domain G. For the details and more general results, see [16] for example. Suppose that there is a (non-trivial) solution $u(\cdot)$. We consider the following problem in this section.

Problem 4. *What is the connection between the behavior of the solution $u(\cdot)$ of the Dirichlet problem, the coëfficients $A(\cdot) = (a_{ij}(\cdot))_{1\le i,j\le n}$ and the data $f(\cdot)$?*

A.D. Aleksandrov provided an answer to this problem under some general conditions.

Theorem 1.3 (Theorem 8 of Aleksandrov [3])**.** *Suppose that there exists a solution $u(\cdot)$ when the coëfficient $A(\cdot)$ is non-negative definite, and $|f(\cdot)|^p$ is integrable for some $p \ge n$ over the subset $G \subset \mathbb{R}^n$, i.e.,*

$$\|f\|_G := \int_G |f(x)|^p dx < \infty\,. \tag{1.32}$$

There exists a bounded continuous function $h(\cdot)$ specified in (1.50) *and L^p-norm $\|\cdot\|_H$ related to m-dimensional hyperplane H specified in* (1.46)*, for some $1 \le m \le n$, such that*

$$\begin{aligned} |u(\cdot)| &< 2h(\cdot)m^{-1+1/m}v_m^{-1/m}\cdot\|f_+\|_H \quad &&\text{in } \{x\in\mathbb{R}^n : u(x) < 0\}\cap G\,,\\ |u(\cdot)| &< 2h(\cdot)m^{-1+1/m}v_m^{-1/m}\cdot\|f_-\|_H \quad &&\text{in } \{x\in\mathbb{R}^n : u(x) > 0\}\cap G\,. \end{aligned} \tag{1.33}$$

Here the constant v_m is the volume of the $m-$dimensional sphere, and $f_\pm(\cdot)$ are the positive and negative part of function $f(\cdot)$, respectively, i.e., $f_+(x) = \max(f(x), 0)$ and $f_-(x) =$

$\max(-f(x), 0)$, $x \in \mathbb{R}^n$. *Consequently, we have*

$$(1.34) \qquad |u(\cdot)| < 2\big(\sup_{y \in G} h(y)\big)\, m^{-1+1/m}\, v_m^{-1/m} \cdot \|f\|_H \qquad \textit{in } G\,.$$

Remark 1.1. The crucial part of the above conditions is that the matrix-valued function $A(\cdot)$ does not have negative eigenvalues; there is no other restrictive condition. It is allowed, in general, that $A(\cdot)$ may depend on the solution $u(\cdot)$ and its derivatives as well.

1.3.1 Extrinsic Geometry

We introduce some geometric objects in order to discuss Aleksandrov's technique of developing Theorem 1.3. Let us fix an integer m , $1 \le m \le n$.

Hyperplanes and Pencils (Sheaves)

Definition 1.2 (Hyperplanes). We define a $m-$dimensional hyperplane H which passes through the origin, i.e.,

$$H := \Big\{ \sum_{i=1}^{m} x_i \mathfrak{n}_i \,:\, \mathfrak{n}_i \in \mathbb{R}^n \,, i = 1, \ldots, m \Big\}\,,$$

for some m unit column vectors $\mathfrak{n}_1 \ldots, \mathfrak{n}_m$ of $\mathbb{R}^n$, such that rank $(\mathfrak{n}_1 \ldots \mathfrak{n}_m) = m$.

For example, if $m = 1$, the corresponding one dimensional hyperplane $H = \{x\,\mathfrak{n} \,:\, x \in \mathbb{R}\}$ is the line in the direction of $\mathfrak{n} \in \mathbb{R}^n$. If $m = n$, then H is the whole space.

Let us define the $n-$dimensional orthonormal basis $\{\, e_i \,;\, i = 1, \ldots, n \,\}$ where e_i is the $(n \times 1)$ vector whose i-th component is one and others are zeros. We may take the above n_i as e_i by rotation of the hyperplane H . By rotation of the coördinates from the original one $(x_1, \ldots, x_n)$ to another one $(y_1, \ldots, y_n)$ we specify the hyperplane H , so that $y_1, \ldots, y_m$ axes lie in H . In fact, we take $(n-m)$ $(n \times 1)$ vectors $\mathfrak{n}_{m+1}, \ldots, \mathfrak{n}_n$ additionally to $(\mathfrak{n}_1, \ldots, \mathfrak{n}_m)$, so that $\operatorname{rank}(\mathfrak{n}_1 \ldots \mathfrak{n}_n) = n$. From the Gram-Schdmit orthogonalization, we obtain the orthonormal basis $(\widetilde{e}_1, \ldots, \widetilde{e}_n\,)$ from $(\mathfrak{n}_1, \ldots, \mathfrak{n}_n)$. The point $x = \sum_{i=1}^n x_i\, e_i \in \mathbb{R}^n$ can be written as $x = \sum_{i=1}^n y_i \widetilde{e}_i \equiv \phi(y_1, \ldots, y_n)$ in terms of this new coördinate $(\widetilde{e}_1, \ldots, \widetilde{e}_n\,)$, where $\phi : \mathbb{R}^n \mapsto \mathbb{R}^n$ is the linear function. Since the i-th element $x_i = \langle x, e_i \rangle = \sum_{j=1}^n y_j \langle \widetilde{e}_j, e_i \rangle$,

$$(1.35) \qquad \frac{\partial x_i}{\partial y_j} = \frac{\partial \phi(y_1, \ldots y_n)}{\partial y_j} = \langle \widetilde{e}_j, e_i \rangle\,, \qquad \frac{\partial^2 u(\phi(y_1, \ldots y_n))}{\partial y_i \partial y_j} = \sum_{k=1}^{n} \sum_{\ell=1}^{n} \frac{\partial^2 u(x)}{\partial x_i \partial x_j} \langle \widetilde{e}_k, e_i \rangle \langle \widetilde{e}_\ell, e_j \rangle\,.$$

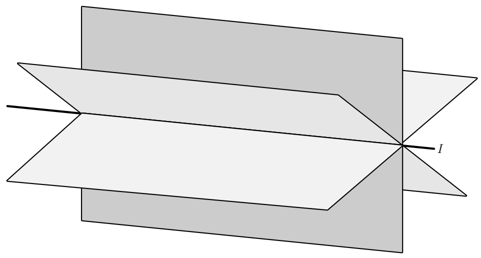

Figure 1.3.1: Pencils of planes with axis I for $m = 2, n = 3$.

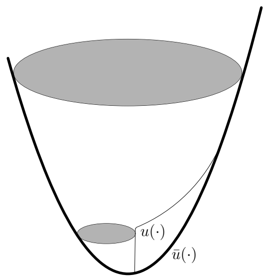

Figure 1.3.2: The convex hull for $z = u(\cdot)$.

Definition 1.3 (Pencils)**.** Let us take an $(m-1)-$dimensional plane I passing through the origin. The set $\mathcal{H}$ of all $m-$dimensional planes passing through the plane I forms a complete pencil (or "sheaf") of planes with axis I. See Figure 1.3.1.

Convex Hulls

We consider the function $u(\cdot)$ as the graph $z = u(x)$ for $x \in \mathbb{R}^n$ in the coördinate of $(x_1, \ldots, x_n, z)$. The convexity of the graph is our key tool.

Definition 1.4 (Convex Hulls)**.** We say that the convex function $\bar{u}(\cdot)$ spanned by a function $u(\cdot)$ is the convex hull of the surface of the graph $z = u(\cdot)$, if $\bar{u}(x)$ is the supremum $\sup_{v \in \mathcal{C}} v(x)$ over the class $\mathcal{C}$ of all convex function $v(\cdot)$ at $x \in \mathbb{R}^n$ smaller than $u(\cdot)$. See Figure 1.3.2.

Total Derivatives

Definition 1.5 (Total second derivatives)**.** Suppose that the function $u : \mathbb{R}^n \mapsto \mathbb{R}$ has the generalized first- and second-order derivatives, ∇u and Δu. Let us define the Hessian matrix $Hu(\cdot) := (\partial^2 u \,/\, \partial x_i \partial x_j)_{1 \le i,j \le n}(\cdot)$ of $u(\cdot)$, and the approximation error $e(\cdot : x_0)$ for $x_0 \in \mathbb{R}^n$:

$$e(x; x_0) := u(x) - \Big(u(x_0) + \langle \nabla u, x - x_0 \rangle + \frac{1}{2} \langle x - x_0 \,, Hu(x_0)\,(x - x_0) \rangle \Big) \,; \quad x \in \mathbb{R}^n \,,$$

where $\langle \cdot, \cdot \rangle$ represents the Euclidean inner product. We say that $u(\cdot)$ has the general first and second total derivatives

$$d\,u = \sum_{i=1}^{n} \frac{\partial u}{\partial x_i}\, d\,x_i \qquad \text{and} \qquad d^2\,u = \sum_{i=1}^{n}\sum_{j=1}^{n} \frac{\partial^2 u}{\partial x_i \partial x_j}\, d\,x_i\, d\,x_j$$

at the point x_0, if the error $e(x)$ converges to zero, as x converges to $x_0 \in \mathbb{R}^n$ in any direction in $\mathbb{R}^n$.

It is known that if the function $u(\cdot)$ satisfies one of the following Condition 1.1, then it has the total second derivative almost everywhere:

Condition 1.1 (Total second derivatives)**.** One of the followings holds:

(I) the function $u(\cdot)$ has the generalized second-order derivatives in any closed subset $D \subset G$ and the n-th power of the second-order derivatives are integrable over D.

(II) the function $u(\cdot)$ is differentiable throughout G, and moreover,

$$\lim_{\substack{x \to y \\ x, y \in \mathbb{R}^n}} \frac{\|\nabla u(x) - \nabla u(y)\|}{\|x - y\|} < \infty\,,$$

at every point $y \in G$ except for a countable subset of $\mathbb{R}^n$, where $\|\cdot\|$ stands for the Euclidean norm of the vector inside.

When the function $u(\cdot)$ has an ordinary second differential almost everywhere, then it has total second derivative.

Normal Image

Let $v(\cdot)$ be a convex function. Given a point $x_0 \in G \subset \mathbb{R}^n$, there exists at least one $(n \times 1)$-vector $p := (p_1, \ldots, p_n)'$ such that the hyperplane defined by $z = v(x_0) + \langle p, x - x_0 \rangle$ supports the graph $z = v(\cdot)$ at the point x_0, where the bracket $\langle \cdot, \cdot \rangle$ stands for the inner product of two $(n \times 1)$-vectors.

Definition 1.6 (Normal Image)**.** The normal image of x_0 with respect to $v(\cdot)$ is defined as the $(n \times 1)$-vector p. That is, the normal image $p(x_0, v)$ of x_0 with respect to $v(\cdot)$ satisfies

$$p(x_0, v) = \big\{ p \in \mathbb{R}^n \,:\, v(x_0) + \langle\, p\,, x - x_0\,\rangle \le v(x) \;\text{ for } x \in G \big\}\,. \tag{1.36}$$

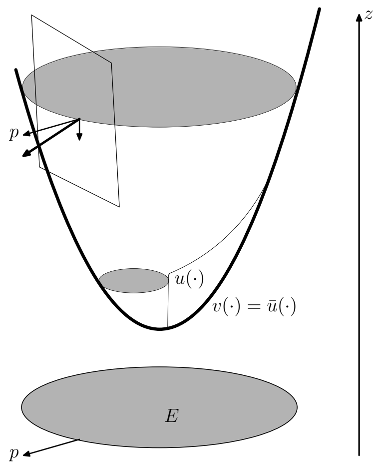

Figure 1.3.3: Normal Image p with respect to the convex function $\bar{u}(\cdot)$.

Further, for a measurable subset $E \subset G$ define the normal image of E with respect to $v(\cdot)$ by $p(E, v) = \cup_{x \in E} p(x, v)$. We define the volume of the normal image as $W(E, v) := Leb(p(E, v))$. See Figure 1.3.3. For the details see A.V. Pogorelov [47] Section VIII.

If the function $u(\cdot)$ satisfies the above Condition 1.1 (I) or (II), then the convex function $\bar{u}(\cdot)$ spanned by $u(\cdot)$ has an absolutely continuous normal image with respect to Lebesgue measure. If a convex function has an absolutely continuous normal image, then the same is true of almost all its non-degenerate projections on the planes in any pencil. The volume $W(E, v)$ of the normal image of the set E with respect to the convex function $v(\cdot)$ is an additive function of the set E. If it is an absolutely continuous normal image, then we may verify that

$$(1.37) \qquad W(E, v) = \int_E \det(V(x))\, dx \quad \text{where} \quad V(\cdot) = \Big(\frac{\partial^2 v}{\partial x_i \partial x_j}(\cdot)\Big)_{1 \le i,j \le n}.$$

Projections on the hyperplane H

Now let us consider the projection G_H of G on the $m-$dimensional hyperplane H. We rotate the coördinate system and rename it, so that the hyperplane H is described by the first $m-$coördinates $x_1, \dots x_m$. Let us write the projection $x_H := (x_1, \dots, x_m)$ of a point $x \in \mathbb{R}^n$ on H. We consider the projection on H of the surface defined by the function as well. We take $a_H(\cdot)$ to mean the principal minor of the matrix of $A(\cdot)$ corresponding to the indices $1, \dots, m$ of the axis which lie in the hyperplane H. If $m = n$, then the principal minor $a_H(\cdot)$ is the determinant $\det\,(A(\cdot))$ of the matrix-valued function $A(\cdot)$.

Define the function $\bar{u}^H : \mathbb{R}^m \mapsto \mathbb{R}$ by

$$\bar{u}^H(x_H) = \inf_{\xi} u(\xi) \tag{1.38}$$

where the infimum is taken over all $\xi \in G \subset \mathbb{R}^n$ for which the projection ξ_H is identical to x_H. The projection on H of the convex hull $z = \bar{u}(x)$ spanned by the surface $z = u(\cdot)$ can be written as $z = v(x)$, where

$$v(x) = v(x_H, x_{m+1}, \ldots, x_n) = \bar{u}^H(x_H); \quad x \in \mathbb{R}^n. \tag{1.39}$$

By definition, $v(\cdot) \leq u(\cdot)$ in $\mathbb{R}^n$.

For almost all hyperplane H the followings hold:

- $\bar{u}^H(\cdot)$ has an absolutely continuous normal image.
- The normal image of the set of points x_H with respect to $\bar{u}^H(\cdot)$, for which $u(\cdot)$ does not have a general second differential, is of zero Lebesgue measure in H.

On the other hand, if $u(\cdot)$ is everywhere twice differentiable, then the above two conditions hold for all H for which $\bar{u}^H(\cdot)$ is non-degenerate.

Definition 1.7. Let M be the set of points $x \in \mathbb{R}^n$ at which the following three conditions are satisfied:

1. the functions $u(\cdot)$ and $v(\cdot)$ satisfy $v(x) = u(x)$.
2. the function $u(\cdot)$ has the total second derivative,
3. the function $v(\cdot)$ is twice differentiable.

Let M_H be the projection of the set M on the hyperplane H and $p(M_H, \bar{u}^H)$ be its normal image with respect to $\bar{u}^H(\cdot)$. By the construction of the normal image and the definition of M we obtain that

$$Leb(\, p(M_H, v)\,) = Leb(\, p(H, \bar{u}^H)\,) = Leb(H \cap p(H, \bar{u}))\,. \tag{1.40}$$

Let K be the $(n+1)-$dimensional cone lying on M. Its projection K_H on H has the normal image $p(K_H, \bar{u}^H)$. We assert that if $v(\cdot)$ is defined on a set of positive Lebesgue measure

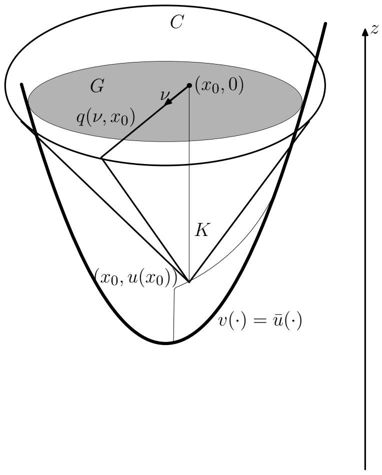

Figure 1.3.4: Auxiliary cone K lying on the convex set C which contains G.

and twice differentiable, modulo a set of Lebesgue measure zero, the intersection $H \cap p(K, v)$ coincides with the normal image $p(K_H, v)$ of K_H:

$$p(K_H, v) = H \cap p(K, v)\,. \tag{1.41}$$

Suppose now that the solution $u(\cdot)$ to the Dirichlet problem (1.31) satisfies $u(x_0) < 0$ for some point $x_0 \in G \subset \mathbb{R}^n$. Let us denote the region where the solution $u(\cdot)$ is negative by $N := \{\, x \in \mathbb{R}^n \,:\, u(x) < 0 \,\} \subset G$. Take an $n-$dimensional convex region $C \subset \mathbb{R}^n$ which contains G and build an $(n+1)-$dimensional cone K with vertex $(x_0, u(x_0))$ lying on it. Thus, $-\,u(x_0)$ is the depth of the cone K from the $n-$dimensional hyperplane $z = 0$ to the vertex. Let the cone be described by the function $u^K(\cdot)$, and define the normal image $p(K, u^K(\cdot))$ of the cone K with respect to the function $u^K(\cdot)$ defining it. Let $S(\nu)$ be the support plane of the region M defined in Definition 1.7 with exterior normal ν and $q(\nu, x_0)$ be the distance from the support plane to the point x_0. The support plane to the cone K passing through the plane $S(\nu)$ forms an angle α with the hyperplane $z = 0$. The angle α is determined by $\tan(\alpha) = |u(x_0)| \,/\, q(\nu, x_0)$; see Figure 1.3.4. Moreover, we observe the relation between the normal images.

$$p(K, \bar{u}^K) \subset p(H, \bar{u}^H)\,. \tag{1.42}$$

With these preparations, we are ready to state the proof of Theorem 1.3. The original lengthy proof is in Aleksandrov's paper [3]. Here, we state it in a simpler and more direct way to emphasize the beautiful connexion between Geometry and Analysis which are sometimes hidden behind the theory of SDEs with bounded coëfficients.

1.3.2 Normal image and the Dirichlet Problem

In this subsection we see that the volume $W(E, v)$ of normal image of set E in (1.37) is utilized for evaluations (1.48) and (1.49) of the solution $u(\cdot)$ of the Dirichlet Problem (1.31). First, it follows from the definition of $v(\cdot)$ in (1.39), of the set M in Definition 1.7, and of the total derivatives, that $v(\cdot) = u(\cdot)$, $d\,v(\cdot) = d\,u(\cdot)$ and $d^2\,v(\cdot) \le d^2\,u(\cdot)$ in M. That is, for the $(m\times m)$ matrix-valued functions

$$U(\cdot) = \Big(\frac{\partial^2 u(\cdot)}{\partial x_i \partial x_j}\Big)_{1\le i,j\le m}, \quad V(\cdot) = \Big(\frac{\partial^2 v(\cdot)}{\partial x_i \partial x_j}\Big)_{1\le i,j\le m},$$

the difference of matrix $(U - V)(\cdot)$ is non-negative definite in M. Moreover, from the convexity of the function $v(\cdot)$ in M, the matrix $V(\cdot)$ is non-negative definite, i.e., $\det(V(\cdot)) \ge 0$. From the definition of $v(\cdot)$ the second derivatives $\partial^2 v \,/\, (\partial x_i \partial x_j)$ for $m < i, j \le n$ are set to be zeroes.

Then, since the matrix-valued function $A(\cdot)$ does not have any negative eigenvalues, there exists the $(n \times n)$ square root matrix $C(\cdot)$ of $A(\cdot)$, such that $a_{ij}(\cdot) = c_i'(\cdot)c_j(\cdot)$ where $C(\cdot) = (c_1(\cdot)\ \dots\ c_n(\cdot)) = (c_{ij}(\cdot))_{1\le i,j\le n}$ with $(n \times 1)$ vector $c_i = (c_{i1}, \dots, c_{in})'$ for $i = 1, \dots n$, we obtain

$$\begin{aligned}\sum_{i=1}^n\sum_{j=1}^n a_{ij}(\cdot)\frac{\partial^2 v(\cdot)}{\partial x_i\,\partial x_j} &= \sum_{i=1}^n\sum_{j=1}^n c_i'(\cdot)c_j(\cdot)\frac{\partial^2 v(\cdot)}{\partial x_i\,\partial x_j} = \sum_{k=1}^n\sum_{i=1}^n\sum_{j=1}^n c_{ik}(\cdot)c_{jk}(\cdot)\frac{\partial^2 v(\cdot)}{\partial x_i\,\partial x_j} \\ &\le \sum_{k=1}^n\sum_{i=1}^n\sum_{j=1}^n c_{ik}(\cdot)c_{jk}(\cdot)\frac{\partial^2 u(\cdot)}{\partial x_i\,\partial x_j} = \sum_{i=1}^n\sum_{j=1}^n a_{ij}(\cdot)\frac{\partial^2 u(\cdot)}{\partial x_i\,\partial x_j} \ \text{ in } M.\end{aligned}$$

Thus, by the use of $\partial^2 v \,/\, (\partial x_i \partial x_j) = 0$ for $m < i, j \le n$ first and then the inequality between the arithmetic and geometric means we obtain

$$\begin{aligned}&\sum_{i=1}^n\sum_{j=1}^n a_{ij}(\cdot)\frac{\partial^2 u}{\partial x_i\partial x_j} \ge \sum_{i=1}^n\sum_{j=1}^n a_{ij}(\cdot)\frac{\partial^2 v}{\partial x_i\partial x_j} = \sum_{i=1}^m\sum_{j=1}^m a_{ij}(\cdot)\frac{\partial^2 v}{\partial x_i\partial x_j} \\ &= \ \mathrm{trace}(\tilde{A}(\cdot)\,V(\cdot)) \ge m\big(\det(\tilde{A}(\cdot)\,V(\cdot))\big)^{1/m} = m(a_H(\cdot)w_H(\cdot))^{1/m} \ge 0 \ \text{ in } M.\end{aligned} \tag{1.43}$$

where $\tilde{A}(\cdot)$ is the first $(m \times m)$ principal matrix of redefined $A(\cdot)$ according to the rotation of the coördinate so that the first m coördinates lie in the hyperplane H, $a_H(\cdot)$ is the principal minor $a_H(\cdot) = \det(\tilde{A}(\cdot)) \geq 0$ of $A(\cdot)$ and

$$(1.44)\qquad \begin{aligned} w_H(x) &:= \det(V(x)) = \det\Big[\Big(\frac{\partial^2 v(x)}{\partial x_i \partial x_j} \Big)_{1 \leq i,j, \leq m} \Big] \\ &= \det\Big[\Big(\frac{\partial^2 \bar{u}^H(x_H)}{\partial x_i \partial x_j} \Big)_{1 \leq i,j, \leq m} \Big] \geq 0 \,; \quad x \in M \,. \end{aligned}$$

From the definition of Dirichlet problem (1.31), it follows that if $u(\cdot)$ is the solution to the Dirichlet problem, then the left-hand of (1.43) is $2f(\cdot)$ and hence it is necessary that $f(\cdot)$ be non-negative in M, i.e., $f(\cdot) = f_+(\cdot)$ in M, where $f_+(\cdot)$ is the positive part of $f(\cdot)$. Thus, we have

$$2f_+(\cdot) \geq m(a_H(\cdot)w_H(\cdot))^{1/m} \text{ in } M \quad \text{or} \quad m^m w_H(\cdot) \leq 2^m\, (a_H(\cdot))^{-1} |f_+(\cdot)|^m \quad \text{in } M \,.$$

We take the supremum over the last $(n-m)$ coördinates $(x_{m+1}, \ldots, x_n)$ of points $x \in G \cap \{x : u(x) < 0\} \equiv G \cap N$ on the right-hand side to obtain

$$m^m w_H(x) \leq 2^m \sup_{\substack{(x_{m+1}, \ldots, x_n) \\ \text{for } x \in G \cap N}} \Big[(a_H(x))^{-1} |f_+(x))|^m \Big] \equiv 2^m \phi(x_H) \,, \quad x \in M \,.$$

Here $\phi(x_H)$ is the supremum of $(a_H(x))^{-1}|f_+(x)|^m$ over the region of x, and it is a measurable function of $x_H = (x_1, \ldots, x_m)$. From (1.44) it follows that both sides do not depend on $(x_{m+1}, \ldots, x_n)$ but on x_H for $x \in M$, so the following inequality holds in the projected region M_H on H:

$$(1.45)\qquad w_H(x_H) = \det\Big[\Big(\frac{\partial^2 \bar{u}^H(x_H)}{\partial x_i \partial x_j} \Big)_{1 \leq i,j, \leq m} \Big] = w_H(x) \leq \Big(\frac{2}{m} \Big)^m \phi(x_H) \,, \quad x \in M_H \,.$$

Then, defining $\|f_+\|_H$ by

$$(1.46)\qquad \begin{aligned} \|f_+\|_H &:= \Big[\int_{G_H} \sup_{\substack{(x_{m+1}, \ldots, x_n) \\ \text{for } x \in G \cap N}} \Big[(a_H(x))^{-1} |f_+(x))|^m \Big] \, d\, x_H \Big]^{1/m} \\ &= \Big[\int_{G_H} \phi(x_H) \, d\, x_H \Big]^{1/m} \,, \end{aligned}$$

and taking the integrals of both sides of (1.45) over the projected region $M_H \subset G_H$, we obtain that

$$(1.47)\qquad \begin{aligned}&\int_{M_H} w_H(x_H)\,d\,x_H \\ &\le \Big(\frac{2}{m}\Big)^m \int_{M_H} \phi(x_H)\,d\,x_H \le \Big(\frac{2}{m}\Big)^m \int_{G_H} \phi(x_H)\,d\,x_H \;\equiv \Big(\frac{2}{m}\Big)^m \|f_+\|_H^m\,.\end{aligned}$$

From (1.37), (1.40), (1.41) and (1.42), the left-hand of (1.47) is

$$(1.48)\qquad \begin{aligned}&\int_{M_H} w(x_H)\,d\,x_H = W(M_H, v) = W(H, \bar{u}^H) \\ &= \int_{p(H,\bar{u}^H)} d\,x_H = \int_{H\cap p(H,\bar{u})} d\,x_H \ge \int_{H\cap p(K,u^K)} d\,x_H = W(K_H, u^K)\,,\end{aligned}$$

and using polar coördinates we compute the right-hand of (1.48) as

$$(1.49)\qquad \begin{aligned}&W(K_H, u^K) \\ &= \int_{S^m}\int_0^{\frac{|u(x_0)|}{q(\nu,x_0)}} r^{m-1}\,d\,r\,d\,\sigma(\nu) = \frac{|u(x_0)|^m}{m}\int_{S^m}\frac{d\,\sigma(\nu)}{(q(\nu,x_0))^m} = \frac{v_m\,\cdot|u(x_0)|^m}{m\,\cdot(h(x_0))^m}\,,\end{aligned}$$

where

$$(1.50)\qquad h(x_0) := \Big(\frac{1}{v_m}\int_{S^m}\frac{d\,\sigma(\nu)}{(q(\nu,x_0))^m}\Big)^{-1/m} > 0\,,$$

and v_m is the volume of $m-$dimensional sphere. Combining these formulae (1.48) and (1.49) with (1.47), we obtain

$$\frac{v_m\,\cdot|m\,u(x_0)|^m}{m\cdot(h(x_0))^m} \le 2^m\|f_+\|_H^m \qquad \text{or} \qquad |u(x_0)| \le \frac{2\|f_+\|_H\cdot h(x_0)}{m^{1-1/m}\,v_m^{1/m}}\,; \quad x_0\in N$$

for the region $N = \{x\,:\,u(x) < 0\,\}$. This is the first part of (1.33). The second inequality of (1.33) is similar if we replace $f(\cdot)$ by $-f(\cdot)$ with necessary modifications. Hence, we obtain (1.34) and complete the proof of Theorem 1.3. □

1.3.3 Application of Aleksandrov's estimate

In this subsection we consider applications of Aleksandrov estimate.

Proof of Proposition 1.2 ; Let us take the solution $u(\cdot)$ to the Dirichlet problem in the Ball

$B_0(r)$ with center zero and radius $r > 0$:

$$
\begin{aligned}
&L^{(k)}u(x) := \sum_{i=1}^{n}\sum_{j=1}^{n} a_{ij}^{(k)}(x)\frac{\partial^2 u(x)}{\partial x_i \partial x_j} = f(x)\,; \quad x \in G_r := B_0(r)\,, \\
&u(x) = 0\,; \quad x \in \{y \in \mathbb{R}^n \,:\, \|y\| = r\} \equiv \partial G_r\,.
\end{aligned} \tag{1.51}
$$

Here $A^{(k)}(\cdot) := (a_{ij}^{(k)}(\cdot))_{1\le i,j\le n}$ is the positive-definite matrix-valued function, which is bounded and continuous; whereas the function $f(\cdot) : \mathbb{R}^n \mapsto \mathbb{R}$ is in $C_0(\mathbb{R}^n)$ with support $\text{supp}\,(f)$ contained in the ball $B_0(R)$. Recall that there exists a unique $C^2(\bar{G}_r)$ solution $u(\cdot)$ in the classical sense, since the domain G_r is bounded and smooth and the datum $f(\cdot)$ is smooth. See [16]. Let $\sigma^{(k)}(\cdot)$ be the square-root of the matrix $a_{ij}^{(k)}(\cdot)$. We consider the Itô process

$$
X(t) = x + \int_0^t \sigma^{(k)}(X(s))\,dW(s)\,, \quad 0 \le t < \infty\,,
$$

where $\sigma^{(k)}(\cdot)$ is bounded and continuous. Let $\mathbb{P}_x^{(k)}$ be the probability measure induced by the above process $X(\cdot)$, and define the first hitting times $\tau_r := \inf\{t \ge 0 \,:\, \|X(t)\| \ge r\}$ of the ball $B(0,r)$ with center origin and radius $r > 0$. Since the matrix $A^{(k)}(\cdot)$ is positive-definite, by the time-change of clock of Brownian motion we obtain $\lim_{r\to\infty}\tau_r = \infty$ $\mathbb{P}_x^k$-a.s. By applying Itô's Lemma to $u(X(t))$, for $0 \le t < T \wedge \tau_r$ for a fixed T and by taking the expectations under the probability measure $\mathbb{P}_x^{(k)}$, we obtain

$$
\mathbb{E}_x^{(k)}\big[u(X(T\wedge\tau_r))\big] = u(x) + \frac{1}{2}\mathbb{E}_x^{(k)}\Big[\int_0^{T\wedge\tau_r} f(X(s))\,ds\Big]\,. \tag{1.52}
$$

Here the local martingale part has zero expectation, since $u(\cdot)$ has the bound (1.34) of Aleksandrov in the region G_r. Moreover, we can approximate both sides of equations by

$$
\begin{aligned}
&\Big|\mathbb{E}_x^{(k)}\Big[\int_0^{T\wedge\tau_r} f(X(s))\,ds - \int_0^{T} f(X(s))\,ds\Big]\Big| \le \mathbb{E}_x^{(k)}\Big[\int_{T\wedge\tau_r}^{T} |f(X(s))|\,ds\Big] \\
&\le \sup_{\substack{x\in\,\text{supp}(f)\\ \subset B(0,R)}} |f(x)|\;\mathbb{E}_x^{(k)}\Big[T - \big(T\wedge\tau_r\big)\Big] \xrightarrow[r\to\infty]{} 0\,,
\end{aligned} \tag{1.53}
$$

and conclude

$$
\begin{aligned}
&\frac{1}{2}\Big|\mathbb{E}_x^{(k)}\Big[\int_0^T f(X(s))\,ds\Big]\Big| \le |u(x)| + \liminf_{r\to\infty}\mathbb{E}_x^{(k)}[|u(X(T\wedge\tau_r))|] \\
&\le 2\liminf_{r\to\infty}\sup_{y\in G_r}|u(y)| \le C\|f\|_{L^n(\mathbb{R}^n)} < \infty\,.
\end{aligned} \tag{1.54}
$$

Since the support supp (f) of the continuous function $f(\cdot)$ is included in the ball $B_0(R)$, $f(\cdot)$ is bounded, and hence we can take the limit as $r \to \infty$ in (1.53). This (1.54) gives (1.22) in Proposition 1.2. $\square$

1.4 Attainability

1.4.1 Hitting the Origin

In this subsection we show that the process with bounded diffusion coëfficients $s(\cdot)$ may visit the origin in finite time with probability one, and hence, under the strong Markov property, the process visits the origin infinitely often. The following construction is due to Bass & Pardoux [9]. Later we generalize their result in Proposition 1.7 in Section 1.6.

The diffusion matrix $s(\cdot)$ in (1.4) has a special characteristic in the allocation of its eigenvalues so that all eigenvalues but the largest of $s(\cdot)$ are small. Let us write a diffusion matrix $\sigma(\cdot)$ which is a piecewise constant function $\sum_{\nu=1}^{m} s_\nu 1_{\mathcal{R}_\nu}(\cdot)$ in each polyhedral region $\mathcal{R}_\nu$, for $\nu = 1, \ldots, m$ with $\cup_{\nu=1}\mathcal{R}_\nu = \mathbb{R}^n$. Here the constant $(n \times n)$ matrices $\{s_\nu,\, \nu = 1 \ldots m\}$ have the decomposition

$$\sigma(\cdot) := \sum_{\nu=1}^{m} s_\nu \mathbf{1}_{\mathcal{R}_\nu}(\cdot)\,, \quad s_\nu s_\nu' := (y_\nu\,,\, B_\nu)\ \mathrm{diag}\ (1, \varepsilon^2, \ldots, \varepsilon^2) \begin{pmatrix} y_\nu' \\ B_\nu' \end{pmatrix}, \tag{1.55}$$

where the fixed $(n \times 1)$ vector $y_\nu \in \mathcal{R}_\nu$ satisfies

$$\|y_\nu\| = 1\,, \quad \frac{|\langle\, x\,,\, y_\nu\,\rangle|^2}{\|x\|^2} \geq 1 - \varepsilon\,; \quad x \in \mathbb{R}_\nu\,, \tag{1.56}$$

and the $(n \times (n-1))$ matrix B_ν consists of $(n-1)$ orthonormal $n-$dimensional vectors orthogonal to each other and orthogonal to y_ν, for $\nu = 1, \ldots m$ for some m. Then

$$\frac{\|x\|^2 \mathrm{trace}\ (s(x)s(x)')}{x's(x)s(x)'x} - 1 \leq \frac{(n-1)\varepsilon^2 + \delta}{1-\delta} < 1\,; \quad x \in \mathbb{R}^n\,. \tag{1.57}$$

This is sufficient for the process X to hit the origin in a finite time. In fact, the norm $\|X(\cdot)\|$ of process $X(\cdot)$ has the dynamics

$$d\,\|X(t)\| = \frac{X(t)'\,s(X(t))}{\|X(t)\|}\,d\widetilde{W}(t) + \frac{\mathrm{trace}\ \big(s(X(t))s'(X(t))\big) - \phi(X(t))}{2\|X(t)\|}dt$$

where

$$\phi(x) := \frac{x' s(x) s(x)' x}{\|x\|^2}\,; \quad x \geq 0\,,$$

and $\widetilde{W}$ is the $n-$dimensional Brownian motion. Let $\tau(0) := \inf\{t \geq 0 \,|\, \|X(t)\| = 0\}$. Introducing the time change $A(t) = \int_0^t \phi(X(u))\, d\, u$ for $t \geq 0$ and looking at the process $S(t) = X(A^{-1}(t))$ with the clock of inverse function A^{-1} of A, we obtain from (1.57) ;

$$d\, S(t) = d\, \widetilde{W}(t) + \frac{C(t)}{2S(t)}\, d\, t\,; \quad 0 \leq t < \infty\,, \qquad \text{where}$$

$$C(t) = \Big(\frac{\text{trace } ss'}{\phi} - 1\Big)(S(t)) = \frac{\|x\|^2 \text{trace } (s(x)s(x)')}{x's(x)s(x)'x} - 1\Big|_{x=S(t)} \leq \frac{(n-1)\varepsilon^2 + \delta}{1-\delta} < 1\,.$$

By the comparison theorem for one-dimensional stochastic differential equations, there exists a Bessel process $R(\cdot)$ of dimension smaller than $2 - \eta$ with $\eta \in (0, 2)$, such that $S(t) \leq R(t)$ for $t \leq \rho_n := \inf\{t \geq 0 : S(t) \leq 1/n \text{ or } R(t) \leq 1/n\}$, $\forall n \geq 1$. Thus, the time-changed process $\{S(t)\}$ hits the origin infinitely often with probability one, and hence so does $\{X(t)\}$. This indicates that the process hits the origin infinitely often, although the standard Brownian motion in dimension greater than or equal to two never does. This phenomenon happens, because the *effective* number of Brownian motion used in $S(t)$ is smaller than 2 by choosing δ and ε small. Note that δ controls the regions $\mathcal{R}_\nu$ and the diffusion coefficient $s(\cdot)$ together. As a summary of the above argument, let us put the following claim about recurrence of processes. This will be generalized in Proposition 1.7 in Section 1.6.

Proposition 1.4. *Under the specification of diffusion coëfficients $s(\cdot)$ with* (1.55), (1.56)*, the process $X(\cdot)$ defined in* (1.4) *may visit the origin infinitely often with probability one. In other words, the origin can be recurrent for multi-dimensional diffusion under appropriate choice* (1.55), (1.56) *of piecewise constant coëfficients.*

1.4.2 Attainability of Submanifolds

The discussion in the previous section brings us to a more general problem of attainability of sub-manifolds $\mathcal{M}$ by the diffusion X in (1.13).

Definition 1.8. A closed set $\mathcal{M} \subset \mathbb{R}^n$ is called *non-attainable* from the initial point x_0 by the process X, if we have $\mathbb{P}_{x_0}(X(t) \in \mathcal{M} \text{ for some } t > 0) = 0$.

It is known that if X is the $n-$dimensional standard Brownian motion, i.e., X follows (1.4) with the coëfficient $s(\cdot)$ being the identity matrix for $n \geq 2$, then $\mathcal{M} = \{0\}$ is non-attainable. Again when $\mathcal{M}$ is $\{0\}$, then Proposition 1.4 says that $\mathcal{M}$ is attainable if the diffusion coëfficient $s(\cdot)$ satisfies (1.55), (1.56). Here is another interesting view on attainability.

Example 1.6 (Collision of Brownian particles)**.** Let $\mathcal{M} := \{x \in \mathbb{R}^n \,:\, x_i = x_j = x_k\}$ for some $1 \leq i < j < k \leq n$. Consider that each component $X_i(\cdot)$ of the process $X(\cdot)$ in (1.13) represents a tiny particle which diffuses on the real line and whose volume is negligible. The question of attainability of this $\mathcal{M}$ by the process X is equivalent to that of collisions among three Brownian particles. If $\mathcal{M}$ is non-attainable, then there is no triple collision among three particles $X_i(\cdot)$, $X_j(\cdot)$ and $X_k(\cdot)$:

$$\mathbb{P}_{x_0}\big(X_i(t) = X_j(t) = X_k(t) \quad \text{for some } t > 0\big) = 0\,. \tag{1.58}$$

We shall study sufficient conditions for (1.58) in Section 1.6 and in the next chapter.

1.4.3 Bounded Continuous Coëfficients

Friedman [13] established theorems on the non-attainability of lower dimensional sub-manifolds of $\mathbb{R}^n$ by non-degenerate diffusions. Assume that the coëfficients $b(\cdot)$ and $\sigma(\cdot)$ satisfy linear growth and Lipschitz conditions, so there exists a unique strong solution. Let $\mathcal{M}$ be a closed $k-$dimensional C^2-manifold in $\mathbb{R}^n$, with $k \leq n-1$. At each point $x \in \mathcal{M}$, let $N_{k+i}(x)$ form a set of linearly independent vectors in $\mathbb{R}^n$ which are *normal to* $\mathcal{M}$ and x. Consider the $(n-k)\times(n-k)$ matrix $\alpha(x) := (\alpha_{ij}(x))$ where

$$\alpha_{ij}(x) = \langle A(x)N_{k+i}(x), N_{k+j}(x)\rangle\,; \quad 1 \leq i,j \leq n-k\,,\ x \in \mathcal{M}\,,$$

and $\langle\cdot,\cdot\rangle$ stands for the inner product in Euclidean space.

Roughly speaking, the strong solution of (1.13) under linear growth and Lipschitz conditions on the coëfficients cannot attain $\mathcal{M}$, if rank $(\alpha(x)) \geq 2$ holds for all $x \in \mathcal{M}$. The rank indicates how wide the orthogonal complement of $\mathcal{M}$ is. If the rank is large, the manifold $\mathcal{M}$ is too *thin* to be attained. The following fundamental Lemma 1.2 based on the partial differential inequality (1.59) leads to Theorem 1.4.

Lemma 1.2. [Friedman [13]] *Consider the process* X *defined in* (1.4) *with coëfficients* $b(\cdot)$ *and* $\sigma(\cdot)$ *satisfying linear growth condition, Lipschitz condition and uniformly elliptic condition.*

Suppose that $\mathcal{M}$ is a compact subset of $\mathbb{R}^n$, and there exists a non-negative solution $u(\cdot) \in C^2(\mathbb{R}^n)$ of the partial differential inequality

$$\mathcal{A}\, u(\cdot) \le \mu\, u(\cdot) \tag{1.59}$$

for some $\mu \ge 0$, outside but near $\mathcal{M}$ with $\lim_{dist(x,\mathcal{M})\downarrow 0, x\notin\mathcal{M}} u(x) = \infty$. Then the compact manifold M is non-attainable for X starting at $x_0 \notin \mathcal{M}$.

Remark 1.2. The above Lemma 1.2 can be generalized in three directions. First, it can be shown not only for the strong solution but also for any unique weak solution. Second, for the differentiability of $u(\cdot)$, if the second derivative is piecewise continuous, the conclusion still holds. Third, if the partial differential inequality (1.59) holds outside but near the manifold $\mathcal{M}$, then the conclusion still holds.

We can construct functions $u(\cdot)$ in (1.59) for different cases, and obtain the following Theorem 1.4.

Theorem 1.4. [Friedman [13]] *Assume that the coëfficients $b(\cdot)$ and $\sigma(\cdot)$ in* (1.13) *satisfy linear growth and Lipschitz conditions. If rank$(\alpha(x)) \ge 3$ for $x \in \mathcal{M}$, then $\mathcal{M}$ is non-attainable by the strong solution X. If rank$(\alpha(x)) \ge 2$, and if either $n-k=2$ or $\alpha(x)$ is non-negative definite for all $x \in \mathcal{M}$ with $|x|$ sufficiently small, then $\mathcal{M}$ is non-attainable by the strong solution X.*

Remark 1.3. Ramasubramanian [48] [49] examined the recurrence and transience of projections of weak solution to (1.13) for *continuous* diffusion coëfficient $\sigma(\cdot)$, showing that any $(n-2)-$dimensional C^2-manifold is not hit. The integral test developed there has the integrand similar to the effective dimension studied in Meyer and Serrin [44], as pointed out by M. Cranston in MathSciNet Mathematical Reviews on the Web. We adopt and generalize their idea for the process with piecewise continuous coëfficients in Section 1.6.

1.5 Uniqueness

In this section we discuss the uniqueness of probability distribution of process X defined in (1.13) in Section 1.2.1. When the diffusion coëfficients are bounded and continuous, and moreover, the Cauchy problem (1.17) has a solution,the uniqueness holds as we have seen in Theorem 1.2 in Section 1.2.2. Now we give partial answers to Problem 2 posed in Section 1.2.2 .

1.5.1 Nearly Constant Coëfficients

When the diffusion coëfficient $\sigma(\cdot)$ is a constant matrix $\bar{\sigma}$, and so is the variance-covariance matrix-valued function $A(\cdot) \equiv \bar{\sigma}\bar{\sigma}'$ in $\mathbb{R}^n$, the martingale problem for $A(\cdot)$ is well-posed, since the probability distribution of $\{ X(t) \}$ defined by

$$X(t) = x_0 + \int_0^t \sigma(X(s))\, dW(s)\,; \quad x_0 \in \mathbb{R}^n\,, \tag{1.60}$$

is uniquely determined by the transformation of probability distribution of Brownian motion $\{W(t)\}$. The uniqueness of martingale problem for bounded measurable coëfficients is preserved, if the diffusion coëfficients $\sigma(\cdot)$ is close to a constant matrix. Stroock & Varadhan show the following result.

Theorem 1.5 (Stroock & Varadhan [55] Theorem 7.1.6)**.** *Let $a(\cdot)$ be a bounded measurable symmetric positive-definite matrix-valued function with*

$$\lambda \|y\|^2 \le y' a(\cdot) y \le \Lambda \|y\|^2\,; \quad x, y \in \mathbb{R}^n\,. \tag{1.61}$$

Suppose that there exist a constant $(n \times n)$ matrix c, a constant $p > 1$ and some $C_n(p, \lambda, \Lambda)$, such that

$$\sup_{x \in \mathbb{R}^n} \|a(x) - c\| \le n^{-2}\, C_n(p, \lambda, \Lambda)\,; \quad p > (n+2)/2\,. \tag{1.62}$$

Then, the martingale problem for no-drift coëfficient and variance-covariance matrix $a(\cdot)$ is well-defined. Here, the constant $C_n(p, \lambda, \Lambda)$ depends only on n, p, λ, and Λ, and satisfies

$$\Big\| \frac{\partial^2 G^T_{(c)} f}{\partial x_i \partial x_j}(t, x) \Big\|_{L^p([0,T]\times\mathbb{R}^n)} \le C_n(p, \lambda, \Lambda)\, \|f\|_{L^p([0,T]\times\mathbb{R}^n)}\,, \tag{1.63}$$

where for $0 \le s < t$, $x, y \in \mathbb{R}^n$,

$$g^{(c)}(s, x\,; t, y) = (2\pi)^{-n/2} [\; det\ (c)]^{-1/2} (t-s)^{-n/2} \exp\Big[-\frac{1}{2}(y-x)'[c\,(t-s)]^{-1}(y-x) \Big]\,, \tag{1.64}$$

and for $f \in C_0^\infty([0,\infty) \times \mathbb{R}^n)$,

$$G^T_{(c)} f(t, x) = \int_t^T du \int_{\mathbb{R}^n} f(u, y)\, g^{(c)}(t\,, x\,;\, u\,, y)\, dy\,; \quad 0 \le t \le T\,. \tag{1.65}$$

The proof of this Theorem 1.5 needs a lengthy derivation of the estimate (1.63) with relation to the theory of singular integrals. This part is shown throughly in Appendix of Stroock & Varadhan [55].

One-dimensional case

When the process is one-dimensional process, i.e., $n = 1$, we can take $p = 2 > (1+1)/2$, the constant $C_1(2, \Lambda, \Lambda) = \Lambda/2$ and obtain

$$\sup_{x \in \mathbb{R}^n} \| a(y) - \Lambda I \| \le \Lambda - \lambda = \frac{2(1 - \lambda\Lambda^{-1})}{C_1(2\,, \Lambda\,, \Lambda)}\,; \tag{1.66}$$

this yields the uniqueness of the process. Thus, we can establish the uniqueness of process with measurable bounded coëfficients for one-dimensional case.

Two-dimensional case

Now consider two-dimensional case. Define a continuous map G_μ from $L^2(\mathbb{R}^n)$ into $C_b(\mathbb{R}^n)$ by

$$(G_\mu f)\,(x) = \int_0^\infty d\,t\,\frac{e^{-\mu t}}{2\pi t}\int_{\mathbb{R}^2} e^{-\|x-y\|^2/(2\,t)} f(y)\,d\,y\,; \quad \mu > 0\,,\ x \in \mathbb{R}^2, \tag{1.67}$$

and define another continuous map K_μ from $L^2(\mathbb{R}^n)$ into $C_b(\mathbb{R}^n)$ by

$$(K_\mu\,f)\,(x) = \Big(\mathcal{A} - \frac{1}{2}\Delta\Big) G_\mu\,f(x)\,; \quad \mu > 0\,,\ x \in \mathbb{R}^2\,. \tag{1.68}$$

Now we observe the following estimate.

Lemma 1.3. *When trace $(a(x)) = 2$, we have*

$$\|K_\mu f\|_{L^2(\mathbb{R}^2)} \le (1-\lambda)^2 \|f\|_{L^2(\mathbb{R}^2)}\,. \tag{1.69}$$

Similarly, we can show that

$$\Big|\mathbb{E}^x \int_0^\infty e^{-\mu t} f(X(t))\,d\,t\Big| \le C\,\|f\|_{L^2}\,; \quad f \in C_0^\infty(\mathbb{R}^2)\,,\ \ \text{supp}\,(f) \subset B_0(r)\,, \tag{1.70}$$

for some constant C only depends on $\mu\,, \lambda\,, \Lambda$ and r. Define a measure $\Gamma(x\,, B\,, C\,) :=$ $\mathbb{E}^x[\int_B \mathbf{1}_C(X(s))\,d\,s]$ for $B \in \mathcal{B}(\mathbb{R}_+)$ and $C \in \mathcal{B}(\mathbb{R}^2)$. It follows from (1.70) that this measure is absolutely continuous with respect to Lebesgue measure, and hence there exists a density $\phi(x,s,y)$ for $s \in [0,\infty)$ and $y \in \mathbb{R}^2$. Using this observation, we can write

$$\mathbb{E}^x\Big[\int_0^\infty e^{-\mu\, s}\, f(X(s))\, d\, s\Big] = \int_0^\infty e^{-\mu\, s}\, f(y)\, \phi(x\,,s\,,y)\, d\, y\, d\, s\,; \quad x \in \mathbb{R}^2\,,\ \mu > 0\,. \tag{1.71}$$

Moreover, by Itô's formula, we observe

$$\mathbb{E}^x\big[e^{-\mu\, t}(G_\mu f)(X(t))\big] - (G_\mu f)(x) = \mathbb{E}\Big[\int_0^t (-\mu\, G_\mu + \mathcal{A}\, G_\mu) f(X(s))\, d\, s\Big]\,; \quad f \in C_b^\infty(\mathbb{R}^2)\,,\ \mu > 0\,,$$

and then letting $t \uparrow \infty$, we obtain

$$\begin{aligned}(G_\mu f)(x) &= -\mathbb{E}\Big[\int_0^\infty e^{-\mu\, s}(-\mu G_\mu + \mathcal{A}\, G_\mu) f(X(s))\, d\, s\Big] \\ &= \mathbb{E}\Big[\int_0^\infty e^{-\mu\, s}(I - K_\mu) f(X(s))\, d\, s\Big]\,,\end{aligned} \tag{1.72}$$

since we have (1.68) and

$$\mu\, G_\mu f(x) = \Big(\frac{1}{2}\Delta + I\Big) f(x)\,; \quad f \in C_b^2(\mathbb{R}^2)\,,\ x \in \mathbb{R}^2\,.$$

Thus, combining (1.71) and (1.72), we obtain

$$\mathbb{E}^x\Big[\int_0^\infty e^{-\mu\, s}\, f(X(s))\, d\, s\Big] = G_\mu \circ (I - K_\mu)^{-1} f(x)\,; \quad f \in C_b^2(\mathbb{R}^2)\,,\ x \in \mathbb{R}^2\,.$$

This implies that the process $X(\cdot)$ is uniquely determined. Thus, the martingale problem for bounded measurable $a(\cdot)$ is well-posed in two dimensions. As a summary, we state the following Proposition 1.5. The above proofs come from [55] and [38].

Proposition 1.5. *For a one-dimensional or two-dimensional process X the martingale problems with bounded measurable strictly elliptic coëfficients is well-posed.*

We use this result to show uniqueness of the process up to the first exit time with the piecewise constant coëfficients in the proof of Theorem 1.9 in Section 1.5.3.

1.5.2 Piecewise Constant Coëfficients

Assume $\mathbb{P}^x$ is a weak solution obtained in Proposition 1.3 to the martingale problem for (1.4). In this section we assume that the diffusion coëfficient is piecewise constant in each one of polyhedra which split the whole space $\mathbb{R}^n$, as we introduced (1.7) in Section 1.1.3. Since the diffusion coëfficient $s(\cdot)$ is discontinuous, the boundary behavior of the process around discontinuities under $\mathbb{P}^x$ is questionable, especially if the initial point x is near the boundary. In the previous Section 1.4 we observed that the process may hit such a point with probability one. The question of uniqueness has to overcome this difficulty.

We discuss how to do this in a special case, namely, with piecewise constant coëfficients in each polyhedral region, based on Bass & Pardoux [9]. We use properties of compact, strongly positive, linear operators, developed by Kreĭn & Rutman [33], which is explained in Section 1.5.3. In Section 1.5.4 we introduce local uniqueness of process and explain how to deal with uniqueness up to first exit times. In Section 1.5.5 we show uniqueness in Theorem 1.12 as an application of Kreĭn-Rutman theorem for a strongly positive linear operator.

An invariant linear operator on a cone

Let us see how we encounter such a linear operator in this section. Define a probability measure $\widetilde{\mathbb{P}}^x$ induced from a process X under $\mathbb{P}^x$ killed when it hits the origin at the first time. Assume temporarily that $\widetilde{\mathbb{P}}^x$ is uniquely determined. We can verify its uniqueness later as in the case of piecewise constant coëfficients defined in (1.7) in Theorem 1.9. Let S be the unit sphere $S := \{x \in \mathbb{R}^n \,|\, \|x\| = 1\}$. In order to look at the uniqueness of process, we embed a Markov chain $\{\, X(\tau(k))/k \,;\, k \ge 1\}$ on S with transition probability density

$$Q(\, x \,,\, d\,y) := \mathbb{P}^x(\, X(\tau(2))/2 \in \, d\,y \,\,;\,\, \tau(2) < \tau(0)\,)\,; \quad x, y \in S\,,$$

where $\tau(r) := \inf\{t \ge 0 \,|\, \|X(t)\| \le r\}$ for $r \in \mathbb{R}_+$. Assume (for simplicity) the diffusion matrix function $A(\cdot) := \sigma(\cdot)\sigma'(\cdot) = (a_{ij}(\cdot))_{1\le i,j\le n}$ is constant on the cones, i.e., $A(x) = A(\|x\|^{-1}x)$ for $x \in \mathbb{R}^n$. For fixed $r > 0$ define $Y(t) := X(t)/r$ and $Z(t) := X(r^{-2}t)$. By applying Itô's formula to $f(Y(\cdot))$ and $f(Z(\cdot))$ with $f \in C^2(\mathbb{R}^n)$, we obtain two processes $M(\cdot)\,, N(\cdot)$ defined by

$$M(t) := f(Y(t)) - f(Y(0)) - \frac{1}{2}\int_0^t \frac{1}{r^2}\sum_{i,j} a_{i,j}(x)\frac{\partial^2 f}{\partial x_i \partial x_j}(Y(s))\,d\,s\,,$$
$$N(t) := f(Z(t)) - f(Z(0)) - \frac{1}{2}\int_0^t \frac{1}{r^2}\sum_{i,j} a_{i,j}(x)\frac{\partial^2 f}{\partial x_i \partial x_j}(Z(s))\,d\,s\,,$$

which are $\mathbb{P}^y-$ martingale and $\mathbb{P}^z-$martingales for $y\,,z\in\mathbb{R}^n$, respectively. The probability distribution of $\{\,Y(t)\,\}$ under $\widetilde{\mathbb{P}}^x$ and the probability distribution of $\{\,Z(t)\,\}$ under $\widetilde{\mathbb{P}}^{x/r}$ solve the same martingale problem up to time $\tau(0)$. Then, by the uniqueness of the process under $\widetilde{\mathbb{P}}^x$, the processes $Y(\cdot)$ under $\widetilde{\mathbb{P}}^x$ and $Z(\cdot)$ under $\widetilde{\mathbb{P}}^{x/r}$ have the same probability distribution if $x\neq 0$. Since hitting distributions are invariant under time changes, we obtain

$$\begin{aligned}Q(\,x/|x|,\,d\,y) &= \widetilde{\mathbb{P}}^{x/|x|}(X(\tau(2))/2\in\,d\,y) = \widetilde{\mathbb{P}}^{x/|x|}(Z(\tau(2))/2\in\,d\,y) = \widetilde{\mathbb{P}}^{x}(Y(\tau(2))/2\in\,d\,y)\\ &= \widetilde{\mathbb{P}}^{x}(X(\tau(2))/(2r)\in\,d\,y) = \mathbb{P}^{x}(X(\tau(2))/(2r)\in\,d\,y\,;\,\tau(2)<\tau(0))\,;\quad r=|x|\,.\end{aligned}$$

By the strong Markov property, we repeat the above computation and obtain

$$Q^n(\,x/|x|\,,d\,y) = \mathbb{P}^x(X(\tau(2^n r))/(2^n r)\in d\,y\,;\tau(2^n r)\leq\tau(0))\,;\quad n\in\mathbb{N}\,.$$

Thus, the transition density function $Q(\cdot\,,d\,y)$ has this scale property. We want to know the behavior of $Q^n(\cdot,d\,y)$ for $n\in\mathbb{N}$ to determine the behavior of $\{\,X(t)\,\}$ around the origin, as we will see it later in Section 1.5.5.

The transition probability density $Q(x,d\,y)$ can be seen as a *nice* positive linear operator on the space $\mathcal{S}$ of functions on $S=\{x\in\mathbb{R}^n\,:\,\|x\|=1\,\}$ by defining

$$Q\,f(x) := \int_S Q(x,dy)f(y)\,;\quad x\in\mathbb{R}^n\,, \tag{1.73}$$

for non-nagative measurable functions f on S. Note that if $f\geq 0$, then $Q\,f\geq 0$. This means that the operator Q leaves a cone $K:=\{f\,:\mathbb{R}^n\mapsto[0,\infty)\}$ *invariant*, i.e., $QK\subset K$. The operator Q satisfies (a) compactness and (b) strong positivity which will be shown in (1.89) and (1.91). Kreĭn & Rutman [33] studied linear operators leaving invariant a cone in a Banach space. They characterized such positive compact operators on *invariant* cones.

1.5.3 Kreĭn-Rutman Theory

Definition 1.9 (Compact operators)**.** Let E be a linear normed space. A linear operator $Q:E\mapsto E$ is said to be compact or completely continuous, if it maps each bounded set into a compact set.

Here is a list of well-known properties of compact operators Q.

- The spectrum of a completely continuous operator consists of zero and of the set of all its eigenvalues. The set of eigenvalues is countable, bounded and has no cluster points except zero.

- The whole spectrum of the operator is contained in the circle with centre at the point zero and radius

$$R := \lim_{n\to\infty} \Big(\sup_{f\neq 0} \frac{||\, Q^n\, f\,||_\infty}{||\, f\,||_\infty} \Big)^{1/n} . \tag{1.74}$$

  Here R is equal to the largest of the moduli of the eigenvalues of the operator Q.

- The resolvent operator $R_\lambda := (Q - \lambda I)^{-1}$ can be expanded in a neighborhood of every nonzero eigenvalue λ_0:

$$R_\lambda = (Q - \lambda I)^{-1} = \sum_{j=-m}^{\infty} (\lambda - \lambda_0)^j \Gamma_j\,, \tag{1.75}$$

  where $\{\Gamma_j\,, j \geq -m\}$ are linear operators for some $m \geq 1$.

- The nonzero eigenvalue λ of the operator $Q : E \mapsto E$ is said to have rank p, if there exists a subspace G of E, invariant with respect to Q, such that

$$(Q - \lambda_0 I)^p f = 0\,, \quad (Q - \lambda_0 I)^q f \neq 0\,; \quad q < p\,,\ f \in G\,. \tag{1.76}$$

  Every nonzero eigenvalue has finite rank.

- If Q is completely continuous, then so is its conjugate operator Q^*. Each eigenvalue has one and the same rank.

- The eigenvalue λ_0 of the operator Q has rank $p = 1$ if and only if

$$Q\phi - \lambda_0 \phi = 0\,, \quad Q^*\psi - \lambda_0 \psi = 0 \tag{1.77}$$

  have no trivial solutions orthogonal to each other.

Definition 1.10 (Cone)**.** A closed semigroup K of a linear normed space E is called a cone if it satisfies the following conditions:

- if $f \in K$, then $\lambda f \in K$ for $\lambda \geq 0$,

- if $f\,, g \in K$, then $f + g \in K$,

- if $f(\neq 0) \in K$, then $-f \notin K$.

Let us write E^* for the conjugate space, consisting of all linear functional of E.

Definition 1.11 (Conjugate cone)**.** Let K be a cone. A subset K^* of linear functionals $\Phi \in E^*$ on the normed linear space E, such that $\Phi(f) \geq 0$ for $f \in K$, is called a conjugate cone.

Definition 1.12 (Invariant operator)**.** A linear operator Q is said to leave a cone K invariant, if $QK \subset K$.

Definition 1.13 (Conjugate operator)**.** A linear operator Q^* on the conjugate space E^* is said to be conjugate of linear operator Q, if $Q^* \Phi(f) := \Phi(Qf)$ for $f \in E$ and $\Phi \in E^*$.

Lemma 1.4 ([33])**.** *If a compact operator Q leaving invariant a cone K has a point spectrum different from zero, then it has a positive eigenvalue ρ not less in modulus than every other eigenvalue, and there exist a function ϕ and a functional Φ such that*

$$Q\,\phi = \rho\,\phi\,, \quad Q^*\Phi = \rho\,\Phi\,. \tag{1.78}$$

As a corollary of Lemma 1.4, we obtain the following.

Lemma 1.5 ([33])**.** *Let Q be a compact operator with $QK \subset K$. Assume that there exist $f \in K$ with $\|f\| = 1$, $c > 0$ and $\ell \in \mathbb{N}$, such that $Q^\ell f \geq c\,f$ element-wise. Then, Q has nonzero eigenvalues. Among those of maximal modulus, there is a positive value $\rho > 0$ not less than $c^{1/\ell}$ such that*

$$Q\,\phi = \rho\,\phi\,, \quad Q^*\,\Phi = \rho\,\Phi\,. \tag{1.79}$$

Definition 1.14 (Strongly positive operator)**.** A linear operator $QK \subset K$ is called strongly positive with respect to K with interior, if for each $f(\neq 0) \in K$, there is a $m_f \in \mathbb{N}$ such that $Q^{m_f} f \in \overset{\circ}{K}$.

Lemma 1.6 ([33])**.** *A strongly positive operator Q satisfies the conditions of the above Lemma 1.5.*

Lemma 1.7 ([33])**.** *Suppose that $QK \subset K$ and that there exist $\rho > 0$ and $f \in \overset{\circ}{K}$, such that $Q\,f = \rho\,f$. Then, $\rho^{-\ell} Q^\ell g$ lies at a positive distance from the frontier of K for $g \in \overset{\circ}{K}$ and $\ell \in \mathbb{N}_0$.*

Lemma 1.8 ([33]). *Suppose that Q is a compact strongly positive operator with respect to a cone K with interior. Then,*

1. *Q has a unique eigenfunction $f \in K$ with $\|f\| = 1$ for largest eignevalue ρ in modulus, i.e., $Q\,f = \rho\,f$; and*
2. *the conjugate operator Q^* has unique strongly positive eigenfunctional $\Phi \in K^*$, i.e., $Q^*\,\Phi = \rho\,\Phi$.*

Conversely, if a compact operator Q has the above properties 1, 2, then it is strongly positive.

Proof. It follows from Lemmata 1.5 and 1.6 that there exist an eigenvalue $\rho > 0$ of maximal modulus, an eigenfunction $f \in K$ and $\Phi \in K^*$ such that $Q\,f = \rho\,f$, $Q^*\,\Phi = \rho\,\Phi$. Note that $f \in \overset{\circ}{K}$ because there exists m_f such that $Q^{m_f}\,f = \lambda^{m_f}\,f \in \overset{\circ}{K}$ or $f = \lambda^{-m_f}\,Q^{m_f}\,f \in \overset{\circ}{K}$. Similarly, we obtain $\Phi > 0$. In fact, $\Phi(f) = \rho^{-m_f}(Q^{*m_f}\Phi)(f) = \rho^{-m_f}\Phi(Q^{m_f}f) > 0$ for some m_f .

Suppose now that $(Q-\rho I)^{m_{f_0}-1}f_0 \neq 0$ and $(Q-\rho I)^{m_{f_0}}f_0 = 0$, for some $f_0 \in \overset{\circ}{K}$ and $m_{f_0} \in \mathbb{N}$. Then, $Q(Q-\rho I)^{m_{f_0}-1}f_0 = \rho\,(Q-\rho I)^{m_{f_0}-1}f_0$. This means that $(Q-\rho I)^{m_{f_0}-1}\,f_0$ is another eigenfunction with respect to the eigenvalue ρ . But then, the linear combination $g_t := t\,f + (1-t)(Q-\rho I)^{m_{f_0}-1}\,f_0$, $t \in \mathbb{R}$, of two eigenfunctions is also eigenfunction with respect to ρ , and it can be on the frontier of cone K with appropriate choice of parameter t , if the eigenfunctions f and $(Q-\rho I)^{m_{f_0}-1}\,f_0$ are not collinear. This is a contradiction to the argument in the previous paragraph that the set of all eigenfunctions of Q with respect to ρ , which especially includes such $g_t \in K \setminus \overset{\circ}{K}$, is inside of K . Thus, f and $(Q-\rho I)^{m_{f_0}-1}\,f_0$ are collinear, i.e., $c\,f = (Q-\rho I)^{m_{f_0}-1}\,f_0$ for some constant $c \neq 0$. But then, if $m_{f_0} \geq 2$, then

$$c\,\Phi(f) = \Phi(c\,f) = \Phi((Q-\rho I)^{m_{f_0}-1}f_0) = (Q^*-\rho I)^{m_{f_0}-1}\Phi(f_0) = 0\,, \tag{1.80}$$

which is impossible for $c \neq 0$. Therefore, $m_{f_0} = 1$ or ρ is a simple eigenvalue.

Assume that there exists an eigenvector $f_1 \in K$ for smaller eigenvalue ρ_1 with $|\rho_1| < \rho$, i.e., $Q\,f_1 = \rho_1\,f_1$. Then, $\rho^{-\ell}\,Q^\ell\,f_1 = (\rho_1/\rho)^\ell\,f_1 \xrightarrow[\ell\to\infty]{} 0$, which is a contradiction to Lemma 1.7. Thus, Q cannot have any eigenvalue smaller than ρ in modulus with eigenfunction lying in K . Since ρ is the maximum eigenvalue in modulus, f is the only eigenfunction lying in K . Similarly, we can show that $\Phi > 0$ is the only eigenfunctional in K^* .

Suppose now that $Q\,f_0 = \rho_0\,f_0$ where $\rho_0 = \rho\,e^{i\theta}$ with $|\rho_0| = \rho$, $0 < \theta < 2\pi$ and $f_0 =$

$f_1 + i f_2 \in E$. Then,we choose some constants c_1 and c_2 so that $f + c_1 f_1 + c_2 f_2 \in K \setminus \overset{\circ}{K}$ and

$\rho^{-\ell} Q^{\ell}(f + c_1 f_1 + c_2 f_2) = f + (c_1 \cos(\ell\theta) + c_2 \sin(\ell\theta)) f_1 + (-c_1 \sin(\ell\theta) + c_2 \cos(\ell\theta)) f_2 \,; \quad \ell \in \mathbb{N}\,,$

and $Q^{m_f} f \in \overset{\circ}{K}$ for some m_f. However, if we choose a subsequence $m_f \leq \ell(k) \uparrow \infty$ for $k \geq 1$ such that $e^{i\ell(k)\theta} \xrightarrow[k\to\infty]{} 1$, then we obtain

$$\lim_{k\to\infty} \rho^{-\ell(k)}\, Q^{\ell(k)}\,(f + c_1 f_1 + c_2 f_2) = f + c_1 f_1 + c_2 f_2 \in K \setminus \overset{\circ}{K}\,,$$

which is a contradiction to Lemma 1.7. Thus, ρ is the largest eigenvalue in modulus.

If $Q\,g = \lambda g$ with $\lambda \neq \rho$, then $\lambda \Phi(g) = \Phi(Q\,g) = Q^* \Phi(g) = \rho \Phi(g)$ implies $\Phi(g) = 0$, since $\lambda \neq \rho$. Let us define $Q_1 := Q - \rho\phi\Phi$. Then, Q_1 has the same eigenvalue as Q:

$$Q_1\, g = Q\, g - \rho\,\phi\,\Phi(g) = Q\, g = \lambda\, g\,.$$

Conversely, if $Q_1 g = \lambda\, g$, then we obtain:

$$\lambda \Phi(g) = \Phi(Q_1\, g) = \Phi(Q\, g - \rho\,\phi\,\Phi(g)) = \Phi(Q\, g) - \rho(\Phi(g)) = Q^*\,\Phi(g) - \rho\,\Phi(g) = 0\,.$$

and then $Q_1\, g = Q\, g - \rho\,\phi\,\Phi(g) = c\,\rho\,\phi - \rho\,\phi\,\Phi(c\, g) = 0$. The eigenvalues of the operator Q_1 lie in the interior of the circle $|\lambda| \leq \rho$, i.e.,

$$\lim_{m\to\infty} \|Q_1^m\|^{1/m} = \rho_1 \leq \rho\,. \tag{1.81}$$

Since $\Phi(Q_1\, f) = 0$, we obtain $Q^n\, f = \rho^n\, \Phi(f)\, \phi + Q_1^n\, f$ and

$$\lim_{m\to\infty} \left|\rho^{-m}\, Q^m\, f - \Phi(f)\, \phi\right| \leq \rho^{-m} |Q_1^m|\, |\phi| = 0\,. \tag{1.82}$$

If $f \in K$, then $\Phi(f) > 0$ and $\Phi(f)\, \phi > 0$. For sufficiently large ℓ, we obtain $\rho^{\ell}\, Q^{\ell}\, f > 0$ and $Q^{\ell}\, f > 0$. Therefore, Q is strongly positive. □

With the same argument as in the last part of the above proof, we obtain the following

Theorem 1.6 (Kreĭn & Rutman[33])**.** *For strongly positive compact operator Q on the invariant cone, there exist a largest eigenvalue $\rho \in (0, \infty)$, a corresponding strictly positive continuous* *eigenfunction* $\phi(\cdot)$, *a strictly positive continuous functional* $\Phi : C(S) \to \mathbb{R}$, *and an operator* Q_1 , *such that* $Q\,\phi(x) = \rho\,\phi(x)$ *for* $x \in \mathbb{R}^n$ *and* ;

(a) *The radius of* Q_1 *is strictly smaller than* ρ , *i.e.,*

$$\limsup_{\ell\to\infty}\Big(\sup_{f\not\equiv 0}\frac{||Q_1^\ell f||_\infty}{||f||_\infty}\Big)^{1/\ell} < \rho\,; \tag{1.83}$$

(b) *Decomposition of* Q

$$Qf(x) = \rho\,\Phi(f)\,\phi(x) + Q_1 f(x)\,; \quad f \in C(S)\,, x \in S\,; \tag{1.84}$$

and hence,

(c) $Q^\ell f(x) = \rho^\ell\,\Phi(f)\,\phi(x) + Q_1^\ell f(x)$ *for* $\ell \in \mathbb{N}$.

1.5.4 Uniqueness up to the Exit Time

The following Theorem 1.7 plays an essential role in showing uniqueness. Using this Theorem 1.7 we can make the (global) uniqueness problem into local uniqueness problems.

Theorem 1.7 (Stroock & Varadhan[55] Theorem 6.6.1)**.** *Suppose that for each* $x_0 \in \mathbb{R}^n$ *there is an open ball* $B_r(x_0)$, *containing* x_0 *for some* $r > 0$, *such that the solution to the martingale problem for the drift coëfficients* $b_{x_0}(\cdot)$ *and variance-covariance* $a_{x_0}(\cdot)$ *is unique, and* $b(\cdot) \equiv b_{x_0}(\cdot)$ *and* $a(\cdot) \equiv a_{x_0}(\cdot)$ *in* $B_r(x_0)$. *Then, the martingale problem for* $b(\cdot)$ *and* $a(\cdot)$ *is unique.*

Let us go back to the martingale problem for piecewise constant coëfficients. The difficulty is caused by the discontinuity of coëfficients at the boundaries of polyhedra. We split the problem into three cases, namely, (i) x_0 is inside of a polyhedron, (ii) x_0 is a nonvertex boundary point, i.e., there exists an integer $n_1(< n)$ and a coördinate system for a neighborhood of x_0 such that $a(x)$ depends only on the first n_1-coördinates of x for x in the neighborhood. (iii) x_0 is a vertex boundary point, i.e., there are no such n_1 and coördinate system but x_0 is in a boundary of the polyhedron.

Let us start with the first case (i) x_0 is inside of a polyhedron.

Theorem 1.8 (Bass and Pardoux [9])**.** *Suppose that* x_0 *is in the interior of a polyhedron. The martingale problem up to the exit time of neighborhood of* x_0 *has a unique solution.*

Proof. Since the initial point x_0 of $X := \{X(t) : 0 \le t < \infty\}$ is in the interior of a polyhedron, the variance covariance function $A(\cdot) = \sigma(\cdot)\sigma(\cdot)'$ is positive-definite constant matrix $\bar{A}$ in a neighborhood of x_0. Let $\bar{\sigma}$ be the square-roof of $\bar{A}$. Then, $\bar{\sigma}^{-1}X(t)$ is an $n-$dimensional Brownian motion and it determines the probability distribution of $X(\cdot)$ uniquely. □

Using the same idea of the above proof and the following Lemmata 1.9 and 1.10, we obtain Theorem 1.9 for the case (ii) x_0 is a nonvertex boundary point.

Lemma 1.9. *Suppose that the solution $\mathbb{P}^{(1)}$ to the martingale problem for no-drift coëfficients and $(n_1 \times n_1)-$variance-covariance function $a_1(\cdot)$ starting at y_0 is unique. Let us define*

$$a_2(\cdot) := \begin{pmatrix} a_1(\cdot) & 0 \\ 0 & I_{n_2} \end{pmatrix}, \tag{1.85}$$

where I_{n_2} is $(n_2 \times n_2)$ identity matrix. Then, the solution to the martingale problem $\mathbb{P}^{(2)}$ for no-drift coëfficients and variance-covariance function $a_2(\cdot)$ starting at (y_0, z_0) is unique, and moreover, the first n_1 coördinate process Y corresponding to $a_1(\cdot)$ and the second n_2 coördinate process Z corresponding to I_{n_2} are independent.

Lemma 1.10 ([9])**.** *Suppose that the variance-covariance function $a(x)$ depends only on the first n_1 coördinates of $x \in \mathbb{R}^n$, and it can be written as*

$$a(\cdot) = \begin{pmatrix} D(\cdot) & F(\cdot) \\ F(\cdot) & G(\cdot) \end{pmatrix}, \tag{1.86}$$

where $D(\cdot)$ is $(n_1 \times n_1)$ matrix-valued function. Assume that the martingale problem for $D(\cdot)$ starting at y_0 has the unique solution. Then, the solution to the martingale problem starting from , (y_0, z_0) is unique for $z_0 \in \mathbb{R}^{(n-n_1)\times(n-n_1)}$.

Theorem 1.9 ([9])**.** *Suppose that x_0 is a nonvertex boundary point. The martingale problem for piecewise constant diffusion coëfficient in each polyhedron, up to the first exit time from the neighborhood of x_0, has a unique solution.*

Proof. It follows from Proposition 1.5 in Section 1.5.1 that uniqueness holds for one-dimensional and two-dimensional process. So suppose that the induction hypothesis is true for the dimensions $1, 2, \ldots, n-1$, and we shall show for the dimension n. Since x_0 is a nonvertex point, we can find a coördinate system so that the variance covariance function $\widetilde{a}(x)$ is a function of only the first $n_1(\leq n-1)$ coördinates of x. Let us denote by $D(\cdot)$ the first $(n_1 \times n_1)$ submatrix of $\widetilde{a}(\cdot)$. By the induction hypothesis, the martingale problem for the variance covariance function $D(\cdot)$ is unique, and hence applying Lemma 1.10, we obtain the uniqueness of the martingale problem for $a(\cdot)$ starting at nonvertex point x_0, up to exit time of neighborhood of x_0 is well-posed. □

1.5.5 An application of Kreĭn-Rutman Theorem

In this section using Theorem 1.6 in the previous Section 1.5.3, we show uniqueness of process with piecewise constant coëfficeints defined by (1.7), when the initial point x_0 is a vertex boundary point. In order to apply Theorem 1.6 we want to show that the operator Q in (1.73) is compact and strongly positive.

First, we verify that the operator Q is compact. A key result is the following Theorem 1.10 by Krylov & Safonov [37].

Theorem 1.10 (Krylov & Safonov [37]). *Let $c(0,1)$ be an $n-$dimensional open cube centered $0 \in \mathbb{R}^n$ of side length 1. Let τ_B be the first exit time of the process $X(\cdot)$ from an open set B. There exists a nondecreasing function $\phi : (0,1)^n \mapsto (0,1)^n$ such that if $B \subset c(0,1)$, $Leb(B) > 0$, and $x \in c(0,1)$, then $\mathbb{P}^x(\tau_B < \tau_{c(0,1)}) \geq \phi(Leb(B))$.*

Moreover, suppose that a measurable function $h(\cdot)$ is bounded in $c(0,1)$. Assume that $\{ h(X(t \wedge \tau_{c(0,1)})) , 0 \leq t < \infty \}$ is a martingale. Then, $h(\cdot)$ is Hölder continuous, i.e., there exist constants $\beta > 0$, $K > 0$ not depending on $h(\cdot)$, such that

$$|h(x) - h(y)| \leq K \, \|h\|_\infty \, \|x - y\|^\beta \, ; \quad x, y \in c(0,1) \, . \tag{1.87}$$

Note that the above Theorem 1.10 can be applied not only for the cube $c(0,1)$ but also the sphere $S = \{x \in \mathbb{R}^n \, : \, \|x\| = 1\}$. Take a function $f(\cdot)$ on S with $\|f\| \leq 1$. Suppose that $x_0 \in S$ and define $\sigma := \inf\{t \geq 0 \, : \, X(t) \in \partial B_{x_0}(1/2)\}$. Recall that by the strong Markov property we can write

$$Q\, f(x) = \widetilde{\mathbb{E}}^x[\widetilde{\mathbb{E}}^{X(\sigma)}[f(\, X(\tau(2)) \, / \, 2\,)]\,] \, ; \quad x \in S \, . \tag{1.88}$$

By Theorem 1.10, there exist $K > 0$ and $\beta > 0$ independent of $f(\cdot)$ such that

$$|\, Q\, f(x) - Q\, f(y) \,| \leq K \|Q\, f\| \, \|x - y\|^\beta \leq K \, \|x - y\|^\beta \, ; \quad x \, , y \in S \, . \tag{1.89}$$

Thus, $Q\,f(\cdot)$ is uniformly continuous with a modulus of continuity independent of $f(\cdot)$. Note that we have used $\|Qf\| \le 1$. By the Ascoli-Arzela theorem, Q is compact.

Second, we verify that the operator Q is strongly positive. Take a continuous function $f(\cdot)$ in S with $f(\cdot) \ge 0$ but not identically zero. Since, $f(\cdot)$ is continuous, there exists a y, $c > 0$ and $\delta \in (0, 1/2)$ such that $f(z) > c$ whenever $\|y - z\| < \delta$. Define a continuous function $\psi(t) = \phi(t)\,\mathbf{1}_{\{0\le t\le 1\}} + t\,y\,\mathbf{1}_{\{1\le t\le 3\}}$ where $\phi : [0,1] \mapsto S$ is continuous with $\phi(0) = x$ and $\phi(1) = y$. Now we apply the following support theorem.

Theorem 1.11 (Stroock & Varadhan [55] [56])**.** *Given any solution $\mathbb{P}^{x_0}$ to the martingale problem, any positive numbers $\varepsilon > 0$, $T > 0$ and a continuous functions $\psi(\cdot) \in C(\mathbb{R}_+)$,*

$$\mathbb{P}^{x_0}\Big(\sup_{0\le t\le T}\Big\|X(t) - x_0 - \int_0^t \psi(s)\,d\,s\Big\| > \varepsilon\Big) < 1\,. \tag{1.90}$$

The support of $\mathbb{P}^{x_0}$ coincides with the class of continuous functions starting at x_0 in the sense of (1.90)*.*

With the application of Theorem 1.11 we obtain

$$Q\,f(x) \ge \widetilde{\mathbb{E}}^x(f(\,X(\tau(2))\,/\,2)\,\mathbf{1}_{\{\|X(\tau(2))/2-y\|<\delta\}}) \ge c\,\mathbb{P}^x(\sup_{0\le s\le t}\|X(s) - \psi(s)\| < \delta) > 0\,. \tag{1.91}$$

This implies that the operator Q is strongly positive.

As an application of Kreĭn and Rutman's Theorem 1.6, we obtain that for any sequence $\{\nu_k\}$ of probability measure on S and for functions $f\,g$ on $\mathcal{S}$ with f, $g \not\equiv 0$.

$$\lim_{k\to\infty}\frac{\int Q^k f(x)\nu_k(dx)}{\int Q^k g(x)\nu_k(dx)} = \lim_{k\to\infty}\frac{\rho^k\Phi(f)\int_S \phi(x)\nu_k(dx) + \int_S Q_1^k f(x)\nu_k(dx)}{\rho^k\Phi(g)\int_S \phi(x)\nu_k(dx) + \int_S Q_1^k g(x)\nu_k(dx)} = \frac{\Phi(f)}{\Phi(g)}\,, \tag{1.92}$$

because

$$|\rho^{-k}\int_S Q_1^k f(x)\nu_k(dx)| \le \rho^{-k}||Q_1^k||_\infty \xrightarrow[k\to\infty]{} 0\,.$$

Let $\tau(r) := \inf\{t \ge 0 \,:\, X(t) \notin B_0(r)\}$, and fix $M > 2$. It is sufficient for the proof of uniqueness of process $X(\cdot)$ to show the uniqueness of resolvent operator R_β, defined by

$$(R_\beta\,h)(x) = \mathbb{E}^x\int_0^{\tau(M)} e^{-\beta t}h(X(t))\,d\,t\,;\quad x \in \mathbb{R}^n \tag{1.93}$$

for $\mathbb{R} \in \beta > 0$, and any bounded measurable $h(\cdot)$. In particular, for $\beta = 0$,

$$\sup_{\|x\|<M} |(R_0 h)(x)| \le \sup_{\|x\|<M} \mathbb{E}^x[\tau(M)] \cdot \sup_{\|x\|<M} |h(x)| \le c\, M^2 \sup_{\|x\|<M} |h(x)| \,. \tag{1.94}$$

This is because, for $\|x\| < M$, by Ito's formula

$$\mathbb{E}^x(\tau(M)) = \lim_{t\to\infty} \mathbb{E}^x(\tau(M) \vee t) \le \frac{1}{\lambda}(\lim_{t\to\infty} \mathbb{E}^x |X(\tau(M) \vee t)|^2 + x^2) \le c\, M^2 \,,$$

for some constant $c > 0$ and the lower bound λ of the eigenvalues of the matrix function $A(\cdot) = \sigma(\cdot)\sigma(\cdot)'$. Thus, R_0 is a bounded operator on the set of bounded functions whose support lies in the ball of radius M. Then for the uniqueness of the process it suffices to show the uniqueness of R_0:

$$(R_0 h)(x) = \widetilde{E}^x \Big[\int_0^{\tau(M)} h(X(t))\, d\, t \Big] + \widetilde{\mathbb{P}}^x(\tau(0) < \tau(M)) \,\cdot I(h) \,, \tag{1.95}$$

where

$$I(h) := \mathbb{E}^0 \Big[\int_0^{\tau(M)} h(X(t))\, d\, t \Big] \,.$$

We discussed the uniqueness of $\widetilde{\mathbb{P}}^x$ for interior points or nonvertex boundary points x in Section 1.5.4, so now we examine uniqueness of $I(h)$. Since the process $X(\cdot)$ cannot stay at the origin on a set of positive Lebesgue measure, it suffices to show uniqueness of $I(h)$ for which $h(\cdot)$ vanishes in a neighborhood of the origin. As in the derivation of (1.92), we obtain the following result.

Lemma 1.11 ([9])**.** *For any bounded measurable function $h : \mathbb{R}^n \to \mathbb{R}$ that is zero in a neighborhood of the origin, the value $I(h)$ depends only on h and $\{\widetilde{\mathbb{P}}^x \,, x \neq 0\}$.*

$$I(h) = \lim_{k\to\infty} \frac{\int_S Q^k f(x) \nu_k(d\, x)}{\int_S Q^k g(x) \nu_k(d\, x)} = \frac{\Phi(f)}{\Phi(g)} = c(f, g) \,, \tag{1.96}$$

where $f : S \mapsto \mathbb{R}$ and $g : S \mapsto \mathbb{R}$ are defined by

$$f(x) := \widetilde{\mathbb{E}}^{\delta\, x} \Big[\int_0^{\tau(M)} h(X(t))\, d\, t \Big] \,, \quad g(x) := \widetilde{\mathbb{P}}^{\delta\, x}(\tau(0) > \tau(M)) \,; \quad x \in \mathbb{R}^n \,,\ 0 < \delta < 1,$$

and the probability distribution $\nu_k(\cdot)$ is defined by $\nu_k(dx) := \mathbb{P}^0(X(\tau(2^{-k}\delta)) \in dx)$.

Finally, we are in a position to state the following result.

Theorem 1.12 (Bass and Pardoux [9]). *If* $\tilde{\mathbb{P}}^x$ *is uniquely determined and* $a(x) = a(|x|^{-1}x)$ *, then for* $x \in \mathbb{R}^n$ *, there is at most one solution* $\mathbb{P}^x$ *to the martingale problem starting at* x *.*

Proof. It suffices to consider initial point x_0 which is a vertex boundary point. By a change of coördinate systems, we may assume $x_0 = 0$. The variance-covariance function $A(\cdot)$ can be written as $A(x) = A(\varepsilon x \,/\, \|x\|)$ if $\|x\| < \varepsilon$. Then, if we define $\widetilde{A}(\cdot) := A(\varepsilon x \,/\, \|x\|)$, then $\widetilde{A}(x) = \widetilde{A}(x \,/\, \|x\|)$. Thus, the variance-covariance function becomes an applicable form of Lemma 1.11. Therefore, the uniqueness for $\mathbb{P}^x$ is obtained. ☐

1.6 Triple Collisions

In this section we consider

$$
\begin{aligned}
&\mathbb{P}_{x_0}\big(X_i(t) = X_j(t) = X_k(t) \quad \text{for some } t \geq 0\big) = 0 \quad \text{or} \\
&\mathbb{P}_{x_0}\big(X_i(t) = X_j(t) = X_k(t) \quad \text{for some } t \geq 0\big) = 1\,; \quad x_0 \in \mathbb{R}^n
\end{aligned} \tag{1.97}
$$

for some $1 \leq i < j < k \leq n$. Put differently, we study conditions on their drift and diffusion coëfficients, under which three Brownian particles moving on the real line can collide at the same time, and conditions under which such "*triple collisions*" never occur. Propositions 1.6 and 1.7 below provide partial answers to these questions.

1.6.1 The Setting

Consider the stochastic integral equation (1.13) with bounded measurable drift $b(\cdot)$ and bounded piecewise continuous diffusion coëfficient

$$
\sigma(\cdot) = \sum_{\nu=1}^{m} \sigma_\nu(\cdot) \mathbf{1}_{\mathcal{R}_\nu}(\cdot) \tag{1.98}
$$

for some partitions $\mathcal{R}_\nu$ with $\cup_{\nu=1}^m \mathcal{R}_\nu = \mathbb{R}^n$, and assume that the matrix-valued functions $\sigma_\nu(\cdot)$, $\nu = 1, \ldots, m$ are uniformly positive-definite. Then, the inverse $\sigma^{-1}(\cdot)$ of the diffusion coëfficient $\sigma(\cdot)$ exists in the sense $\sigma^{-1}(\cdot) = \sum_{\nu=1}^m \sigma_\nu^{-1}(\cdot)\mathbf{1}_{\mathcal{R}_\nu}$. As usual, a *weak solution* of this equation consists of a probability space $(\Omega, \mathcal{F}, \mathbb{P})$; a filtration $\{\mathcal{F}_t\,,\, 0 \leq t < \infty\}$ of sub-σ-fields of $\mathcal{F}$ which satisfies the *usual conditions* of right-continuity and augmentation by the $\mathbb{P}-$negligible sets in $\mathcal{F}$; and two adapted, n-dimensional processes on this space $X(\cdot)$, $W(\cdot)$ on this space, such that $W(\cdot)$ is Brownian motion and (1.13) is satisfied $\mathbb{P}-$almost surely. The concept of uniqueness associated with this notion of solvability, is that of *uniqueness in distribution* for the process $X(\cdot)$.

1.6.2 Removal of Drift

We start by observing that the piecewise continuous drift has some effects on the probabilities (1.97). In fact, define an $n-$dimensional process $\xi(\cdot)$ by

$$\xi(t) := \sigma^{-1}(X(t))b(X(t))\,, \qquad 0 \le t < \infty\,.$$

By the nature of the functions $b(\cdot)$ and $\sigma(\cdot)$ in , the mapping $t \mapsto \xi(t)$ is right-continuous or left-continuous on each boundary $\partial\mathcal{R}_{\mathfrak{p}(X(t))}$ at time t, deterministically, according to the position $\mathcal{R}_{\mathfrak{p}(X(t-))}$ of $X(t-)$. Then, although the sample path of n-dimensional process $\xi(\cdot)$ is not entirely right-continuous or left-continuous, it *is* progressively measurable. Moreover, $\xi(\cdot)$ is bounded, so the exponential process

$$\eta(t) = \exp\left[-\int_0^t \langle \xi(u), d\,W(u)\rangle - \frac{1}{2}\int_0^t \|\xi(s)\|^2 d\,u \right]; \quad 0 \le t < \infty \tag{1.99}$$

is a continuous martingale, where $\|x\|^2 := \sum_{j=1}^n x_j^2$, $x \in \mathbb{R}^n$ stands for n-dimensional Euclidean norm and the bracket $\langle x, y\rangle := \sum_{j=1}^n x_j y_j$ is the inner product of two vectors $x, y \in \mathbb{R}^n$. By Girsanov's theorem

$$\widetilde{W}(t) := W(t) + \int_0^t \sigma^{-1}(X(u))b(X(u))d\,u\,,\, \mathcal{F}_t\,; \quad 0 \le t < \infty$$

is an n-dimensional standard Brownian motion under the new probability measure $\mathbb{Q}$, *locally* equivalent to $\mathbb{P}$, that satisfies

$$\mathbb{Q}_{x_0}(C) = \mathbb{E}^{\mathbb{P}_{x_0}}\big(\eta(T)1_C\big)\,; \quad C \in \mathcal{F}_T\,, \quad 0 \le T < \infty\,. \tag{1.100}$$

Let us define an increasing family

$$C_T := \{X_i(t) = X_j(t) = X_k(t)\,, \quad \text{for some } t \in [0,T]\,\}\,; \quad T \ge 0$$

of events. If we *knew*

$$\mathbb{Q}_{x_0}(X_i(t) = X_j(t) = X_k(t) \ \text{ for some } t > 0) = 0\,, \tag{1.101}$$

then we would obtain $0 = \mathbb{Q}_{x_0}(C_\ell) = \mathbb{P}_{x_0}(C_\ell)$ for $\ell \geq 1$, and so

$$\begin{aligned} &\mathbb{P}_{x_0}(X_i(t) = X_j(t) = X_k(t) \ \text{ for some } t > 0) = \mathbb{P}_{x_0}(\cup_{\ell=1}^{\infty} C_\ell) \\ &= \lim_{\ell \to \infty} \mathbb{P}_{x_0}(C_\ell) = 0\,. \end{aligned} \tag{1.102}$$

Thus, in order to evaluate the probability of absence of triple collision in (1.97), let us consider the case of $b(\cdot) \equiv 0$ in (1.13), namely

$$X(t) \;=\; x_0 + \int_0^t \sigma\big(X(s)\big)\, d\widetilde{W}(s)\,, \qquad 0 \leq t < \infty\,. \tag{1.103}$$

under the new probability measure $\mathbb{Q}_{x_0}$. The infinitesimal generator $\mathcal{A}$ of this process, defined on the space $C^2(\mathbb{R}^n;\mathbb{R})$ of twice continuously differentiable functions $\varphi : \mathbb{R}^n \to \mathbb{R}$, is given as

$$\mathcal{A}\varphi(x) := \frac{1}{2} \sum_{i,k=1}^{n} \; a_{ik}(x)\, \frac{\partial^2}{\partial x_i \partial x_k}\, [\varphi(x)]\,; \quad \varphi \in C^2(\mathbb{R}^n;\mathbb{R}), \tag{1.104}$$

where

$$a_{ik}(x) := \sum_{j=1}^{n} \sigma_{ij}(x)\, \sigma_{kj}(x)\,, \quad A(x) := \big\{a_{ij}(x)\big\}_{1 \leq i,j \leq n}\,; \quad x \in \mathbb{R}^n. \tag{1.105}$$

Here $\sigma_{ij}(\cdot)$ is the (i,j)-th element of the matrix-valued function $\sigma(\cdot)$ for $1 \leq i,j \leq n$. By the assumption of uniform positive-definiteness on the matrices $\{\sigma_\nu\}(\cdot)$, $\nu = 1, \cdots, m$ in (1.98), the operator $\mathcal{A}$ is uniformly elliptic. As is well known, existence (respectively, uniqueness) of a weak solution to the stochastic integral equation (1.103), is equivalent to the solvability (respectively, well-posedness) of the martingale problem associated with the operator $\mathcal{A}$.

1.6.3 Comparison with Bessel processes

Without loss of generality we start from the case $i = 1$, $j = 2$, $k = 3$ in (1.97). Let us define $(n \times 1)$ vectors d_1, d_2, d_3 to extract the *information* of the diffusion matrix $\sigma(\cdot)$ on (X_1, X_2, X_3), namely

$$d_1 := (1, -1, 0, \cdots, 0)', \; d_2 := (0, 1, -1, 0, \cdots, 0)', \; d_3 := (-1, 0, 1, 0, \cdots, 0)',$$

where the superscript $'$ stands for transposition. Define the $(n \times 3)$-matrix $D = (d_1, d_2, d_3)$ for notational simplicity. The cases we consider in (1.97) for $i = 1, j = 2, k = 3$ are equivalent to

$$\begin{aligned}
&\mathbb{P}_{x_0}\Big(s^2(X(t)) = 0\,, \quad \text{for some } t \geq 0\Big) = 0 \quad \text{and} \\
&\mathbb{P}_{x_0}\Big(s^2(X(t)) = 0\,, \quad \text{for some } t \geq 0\Big) = 1\,; \quad x_0 \in \mathbb{R}^n\,,
\end{aligned}$$

where the continuous function $s^2 : \mathbb{R}^n \to \mathbb{R}_+$ defined as

$$\begin{aligned}
s^2(x) &:= (x_1 - x_2)^2 + (x_2 - x_3)^2 + (x_3 - x_1)^2 \\
&= d_1' x x' d_1 + d_2' x x' d_2 + d_3' x x' d_3 = x' D D' x\,; \quad x \in \mathbb{R}^n
\end{aligned} \tag{1.106}$$

measures the sum of squared distances for the three particles we are interested in. Thus, it suffices to study the behavior of the continuous, non-negative process $\{s^2(X(t)); 0 \leq t < \infty\}$ around its zero set

$$\mathcal{Z} := \{x \in \mathbb{R}^n : s(x) = 0\}. \tag{1.107}$$

Let us define the following positive, piecewise continuous functions $Q(\cdot)\,,\ R^{(0)}(\cdot)$ computed from the variance-covariance matrix $A(\cdot) = \sigma(\cdot)\sigma(\cdot)'$:

$$\begin{aligned}
R^{(0)}(x) &:= \frac{\text{trace}(D'A(x)D) \cdot x'DD'x}{x'DD'A(x)DD'x} = \frac{\text{trace}(D'A(x)D)}{Q(x)}\,, \text{ where} \\
Q(x) &:= \frac{x'DD'A(x)DD'x}{x'DD'x}\,; \qquad x \in \mathbb{R}^n \setminus \mathcal{Z}\,.
\end{aligned} \tag{1.108}$$

Under the new probability measure $\mathbb{Q}_{x_0}$ of (1.100) the process $s(X(\cdot))$ is a semimartingale with decomposition $d\,s(X(t)) = h^{(0)}(X(t))d\,t + d\,\Theta(t)$, where

$$\begin{aligned}
h^{(0)}(x) &:= \frac{1}{2\,s^3(x)}\left(s^2(x)\sum_{i=1}^3 d_i'\sigma(x)\sigma(x)'d_i - \Big|\Big|\sum_{i=1}^3 \sigma(x)'d_i d_i' x\Big|\Big|^2\right) \\
&= \frac{x'DD'x \cdot \ \text{trace}\ (D'A(x)D) - x'DD'A(x)DD'x}{2(x'DD'x)^{3/2}} \\
&= \frac{(R^{(0)}(x) - 1)Q(x)}{2s(x)}\,; \quad x \in \mathbb{R}^n \setminus \mathcal{Z}\,,
\end{aligned} \tag{1.109}$$

$$\widetilde{\Theta}(t) := \int_0^t \left(\sum_{i=1}^3 \frac{\sigma'(X(\tau)) d_i d_i' X(\tau)}{s(X(\tau))} \right) d\widetilde{W}(\tau)\,,$$
$$\langle \widetilde{\Theta} \rangle(t) = \int_0^t \frac{x' DD' A(x) DD' x}{x' DD' x} \bigg|_{x = X(\tau)} d\tau = \int_0^t Q(X(\tau)) d\tau\,; \quad 0 \le t < \infty\,.$$

respectively.

Now define the stopping time $\Lambda_u := \inf\{t \geq 0 : \langle\widetilde{\Theta}\rangle\,(t) \geq u\}$, and note that we have

$$\mathfrak{s}(u) := s(X(\Lambda_u)) = s(x_0) + \int_0^{\Lambda_u} h^{(0)}(X(t)) d\,t + \widetilde{B}(u)\,; \quad 0 \le u < \infty\,,$$

where $\widetilde{B}(u) := \widetilde{\Theta}(\Lambda_u)\,,\ 0 \le u < \infty$ is a standard Brownian motion, by the Dambis-Dubins-Schwartz theorem of time-change for martingales. Thus, with $\mathfrak{d}(u) := R^{(0)}(X(\Lambda_u))$ we can write

$$d\,\mathfrak{s}(u) = \frac{\mathfrak{d}(u) - 1}{2\,\mathfrak{s}(u)} + d\,\widetilde{B}(u)\,; \quad 0 \le u < \infty\,, \tag{1.110}$$

because

$$h^{(0)}\big(X(\Lambda_u)\big)\,\Lambda_u' \;=\; \frac{[R^{(0)}(X(\Lambda_u)) - 1]\, Q(X(\Lambda_u))}{2\,s(X(\Lambda_u))} \cdot \frac{1}{Q(X(\Lambda_u))} \;=\; \frac{\mathfrak{d}(u) - 1}{2\,\mathfrak{s}(u)}$$

The dynamics of the process $\mathfrak{s}(\cdot)$ are therefore comparable to those of the $\delta-$dimensional Bessel process, namely

$$d\,\mathfrak{r}(u) = \frac{\delta - 1}{2\,\mathfrak{r}(u)}\,d\,u + d\,\widetilde{B}(u)\,; \quad 0 \le u < \infty\,.$$

By a comparison argument similar to Ikeda & Watanabe [24] and Exercise 5.2.19 of Karatzas & Shreve [29], we obtain the following result.

Lemma 1.12. *Suppose that* $x_0 \in \mathbb{R}^n \setminus \mathcal{Z}$. *If* $\underline{\mathfrak{d}} := \operatorname{essinf} \inf_{0 \le t < \infty} \mathfrak{d}(t) \ge 2$, *then*

$$\mathbb{Q}_{x_0}\big(\,\mathfrak{s}(t) > 0\,, \quad \text{for some } t \ge 0\big) = 0\,. \tag{1.111}$$

If $\bar{\mathfrak{d}} := \operatorname{essup} \sup_{0 \le t < \infty} \mathfrak{d}(t) < 2$, *then*

$$\mathbb{Q}_{x_0}\big(\,\mathfrak{s}(t) = 0\,, \quad \text{for some } t \ge 0\big) = 1\,; \tag{1.112}$$

and we have the following estimate

$$\mathbb{Q}_{x_0}\big(\mathfrak{s}(t)=0\,,\quad \text{for some } t\in[0,T]\big)\geq 1-\kappa(T\,;s(x_0),\bar{\mathfrak{d}}) \tag{1.113}$$

where $\kappa(\cdot\,;y,\delta)$ *is the tail probability of hitting-time distribution of Bessel process with dimension* $\delta\in(0,2)$ *starting at* $y>0$ *:*

$$\kappa(T\,;y,\delta):=\int_T^\infty \frac{1}{t\,\Gamma(\,\delta\,)}\Big(\frac{y^2}{2\,t}\Big)^{\delta} e^{-\frac{y^2}{2\,t}}\,d\,t\,;\quad 0\leq T<\infty\,,\quad y>0. \tag{1.114}$$

Proof of Lemma 1.12. From the assumption $x_0\in\mathbb{R}^n\setminus\mathcal{Z}$ where the zero set $\mathcal{Z}$ is defined in (1.107), it follows that $\mathfrak{s}(0)=s(X(\Lambda_0))>0$ and there exists an integer m_0 such that $m_0^{-1}<\mathfrak{s}(0)<m_0$.

Let us consider the case $\bar{\mathfrak{d}}:=\operatorname{essup}\sup_{0\leq t<\infty}\mathfrak{d}(t)<2$ for (1.112). Define two continuous functions $b_1(x):=(\bar{\mathfrak{d}}-1)\,/\,(2x)$ and $b_2(x):=\bar{\mathfrak{d}}\,/\,(4x)$ for $x\in(0,\infty)$. If $\bar{\mathfrak{d}}<2$, then $b_1(\cdot)<b_2(\cdot)$ in $(0,\infty)$. For each integer $m(\geq m_0)$, there exists a non-increasing Lipschitz continuous function $f_m(\cdot):=(b_1(\cdot)+b_2(\cdot))\,/\,2$ with Lipschitz coëfficient $K_m:=\max_{x\in[m^{-1},m]}|b_2'(x)|$, such that $b_1(\cdot)\leq f_m(\cdot)\leq b_2(\cdot)$ in $[m^{-1},m]$.

Now take a strictly decreasing sequence $\{a_n\}_{n=0}^\infty\subset(0,1]$ with $a_0=1$, $\lim_{n\to\infty}a_n=0$ and $\int_{(a_n,a_{n-1})}u^{-2}\,d\,u=n$ for every $n\geq 1$. For each $n\geq1$, there exists a continuous function $\rho_n(\cdot)$ on $\mathbb{R}$ with support in $(a_n\,,a_{n-1})$, so that $0\leq\rho_n(x)\leq 2(nx^2)^{-1}$ holds for every $x>0$ and $\int_{(a_n,a_{n-1})}\rho_n(x)\,d\,x=1$. Then the function $\psi_n(x):=\int_0^{|x|}\int_0^y\rho_n(u)\,d\,u\,d\,y\,;\ x\in\mathbb{R}$ is even and twice continuous differentiable with $|\psi_n'(x)|\leq 1$ and $\lim_{n\to\infty}\psi_n(x)=|x|$ for $x\in\mathbb{R}$. Define $\varphi_n(x):=\psi_n(x)\cdot\mathbf{1}_{(0,\infty)}(x)\,;\ x\in\mathbb{R}\,,\ n\geq1$.

Recall that with $R^{(0)}(X(\Lambda.))=\mathfrak{d}(\cdot)$ and $s(X(\Lambda.))=\mathfrak{s}(\cdot)$ we obtained (1.110):

$$\mathfrak{s}(t)=\mathfrak{s}(0)+\int_0^t\frac{\mathfrak{d}(u)-1}{2\mathfrak{s}(u)}\,d\,u\,+\widetilde{B}(t)\,;\quad 0\leq t<\infty.$$

Define an auxiliary Bessel process $\mathfrak{r}(\cdot)$ with dimension $(\bar{\mathfrak{d}}+2)\,/\,2\ (<2)$ starting at $\mathfrak{s}(0)$:

$$\mathfrak{r}(t):=\mathfrak{s}(0)+\int_0^t b_2(\mathfrak{r}(u))\,d\,u\,+\widetilde{B}(t)\,;\quad 0\leq t<\infty. \tag{1.115}$$

Consider also the increasing sequence of stopping times

$$\tau_m := \inf\{t \geq 0 \,:\, \max[\mathfrak{s}(t), \mathfrak{r}(t)] \geq m \text{ or } \min[\mathfrak{s}(t), \mathfrak{r}(t)] \leq m^{-1}\}$$

for $m_0 \leq m < \infty$, and $\tau := \inf\{t \geq 0 : \mathfrak{r}(t) = 0\}$. By the property of the Bessel process with dimension strictly less than 2, the Bessel process $\mathfrak{r}(\cdot)$ attains the origin within finite time: $\tau < \infty$ holds a.s.

By combining the properties of $\varphi_n(\cdot)$, $b_1(\cdot), b_2(\cdot)$ and $f_m(\cdot)$, we can verify that the difference $\Delta. := \mathfrak{s}(\cdot) - \mathfrak{r}(\cdot)$ is a continuous process which satisfies

$$\begin{aligned}
&\varphi_n(\Delta_t) \leq \int_0^t \varphi_n'(\Delta_u)\big(b_1(\mathfrak{s}(u)) - b_2(\mathfrak{r}(u))\big)\,d\,u \\
&\leq \int_0^t \varphi_n'(\Delta_u)\left(f_m(\mathfrak{s}(u)) - f_m(\mathfrak{r}(u))\right)d\,u \leq \int_0^t \varphi_n'(\Delta_u) K_m(\mathfrak{s}(u) - \mathfrak{r}(u))^+\,d\,u \\
&\leq K_m \int_0^t (\Delta_u)^+\,d\,u\,; \quad 0 \leq t \leq \tau_m\,.
\end{aligned}$$

Letting $n \to \infty$ we obtain $(\Delta_t)^+ \leq K_m \int_0^t (\Delta_u)^+\,d\,u$ for $0 \leq t \leq \tau_m$. Since the difference $\Delta.$ has continuous paths a.s., the Gronwall inequality gives $\Delta. \leq 0$ in $[0, \tau_m]$ a.s. for $m \geq m_0$.

Now by the construction of the stopping times $\{\tau_m\}$ and τ we obtain $\widetilde{\tau} := \lim_{m\to\infty} \tau_m \leq \tau < \infty$. From the continuity of sample paths of $\mathfrak{s}(\cdot), \mathfrak{r}(\cdot)$ in $[0, \infty)$, it follows that

$$\mathfrak{s}(\widetilde{\tau}) = \lim_{t\to\widetilde{\tau}} \mathfrak{s}(t) \leq \lim_{t\to\widetilde{\tau}} \mathfrak{r}(t) = \mathfrak{r}(\widetilde{\tau}) \quad \text{and} \quad \max[\mathfrak{s}(\widetilde{\tau}), \mathfrak{r}(\widetilde{\tau})] < \infty \;\; a.s. \tag{1.116}$$

If $\mathfrak{s}(\widetilde{\tau}) > 0$, then by the definition of $\{\tau_m\}$ we obtain a contradiction $0 = \mathfrak{r}(\widetilde{\tau}) \geq \mathfrak{s}(\widetilde{\tau})$. Therefore, $\mathfrak{s}(\widetilde{\tau}) = 0$ and $\mathfrak{s}(t) \leq \mathfrak{r}(t)$ for $0 \leq t \leq \widetilde{\tau}$. Therefore, for $\bar{\mathfrak{d}} = \operatorname{essup}\sup_{0\leq t<\infty} \mathfrak{d}(t) < 2$ we conclude

$$\mathbb{Q}_{x_0}\big(\, s(X(t)) = 0 \text{ for some } t > 0\big) = \mathbb{Q}_{x_0}\big(\,\mathfrak{s}(t) = 0 \text{ for some } t \geq 0\big) = 1\,.$$

By the strong Markov property of the process $X(\cdot)$ under $\mathbb{Q}$, we obtain

$$1 = \mathbb{Q}_{x_0}\big(s(X(t)) = 0 \quad \text{i.o. }\big) = \mathbb{Q}_{x_0}\big(\,\mathfrak{s}(t) = 0 \quad \text{i.o. }\big)\,.$$

This gives (1.112) of Lemma 1.12. Moreover, by the formula of the first hitting-time probability density function for the Bessel process with dimension $\bar{\mathfrak{d}}$ in Elworthy et al. [11] and Göing-

Jaeschke & Yor [17] we obtain

$$\begin{aligned}&\mathbb{Q}_{x_0}\big(\,\mathfrak{s}(t)=0\,,\quad \text{for some } t\in(0,T]\,\big)\\ \geq\,&\mathbb{Q}_{x_0}\big(\,\mathfrak{r}(t)=0\,,\quad \text{for some } t\in(0,T]\,\big)=1-\kappa(T;s(x_0),\bar{\mathfrak{d}})\end{aligned}$$

where the tail probability distribution function $\kappa(\cdot\,;\cdot,\cdot)$ is defined in (1.114). This gives (1.113) of Lemma 1.12.

Next consider the case of $\underline{\mathfrak{d}} := \operatorname{essinf}\inf_{0\leq t<\infty}\mathfrak{d}(t)\geq 2$. Define $b_3(x) := (\underline{\mathfrak{d}}-1)\,/\,(2x)$ and $b_4(x) := \underline{\mathfrak{d}}\,/\,(4x)$ for $x\in(0,\infty)$. We follow the similar course as in the previous case, using $b_3(\cdot)\,,b_4(\cdot)$ and defining a non-increasing Lipschitz continuous function $g_m(\cdot) := (b_3(\cdot)+b_4(\cdot))\,/\,2$ with Lipschitz coëfficient $L_m := \max_{x\in[m^{-1},m]}|b_3'(x)|$, rather than using $b_1(\cdot)\,,b_2(\cdot)\,,\ f_m(\cdot)$ and K_m. Thus, we obtain the reverse inequality $\mathfrak{r}(\cdot)\leq\mathfrak{s}(\cdot)$ in $[0,\tau_m]$ a.s. where $\mathfrak{r}(\cdot)$ in (1.115) is now redefined as the Bessel process with dimension $(\underline{\mathfrak{d}}+2)\,/\,2\ (\geq 2)$ starting at $\mathfrak{s}(0)$:

$$\mathfrak{r}(t)=\mathfrak{s}(0)+\int_0^t b_4(\mathfrak{r}(u))\,d\,u\,+\widetilde{B}(t)\,;\quad 0\leq t<\infty\,.$$

By the path property of Bessel process of dimension greater than or equal to 2, the process $\mathfrak{r}(\cdot)$ never attains the origin a.s. , i.e., $\mathfrak{r}(\cdot)>0$ in $[0,\infty)$.

If $\widetilde{\tau}=\lim_{m\to\infty}\tau_m<\infty$, then like (1.116) but now we obtain $\mathfrak{s}(\widetilde{\tau})\geq\mathfrak{r}(\widetilde{\tau})>0$ and $\max[\mathfrak{s}(\widetilde{\tau})\,,\mathfrak{r}(\widetilde{\tau})]<\infty$ a.s. From the construction of $\{\tau_m\}$ a contradiction follows: $0=\mathfrak{s}(\widetilde{\tau})>0$. Therefore, $\mathbb{Q}_{x_0}(\mathfrak{s}(t)>0\ \text{ for } 0\leq t<\infty\,)=1$. This gives (1.111) of Lemma 1.12 for the case of $\underline{\mathfrak{d}}\geq 2$. $\square$

The tail probability (1.114) decreases to zero, as $T\to\infty$, in the order of $T^{-\delta}$. Combining this lemma and the results from Section 1.6.2 with the definition $\mathfrak{d}(\cdot)=R^{(0)}(X(\Lambda.))$, we immediately obtain the following Proposition 1.

Proposition 1.6. *Suppose that the matrices* $\sigma_\nu(\cdot)$, $\nu=1,\cdots,m$ *in* (1.98) *are uniformly bounded and positive-definite, and satisfy the following condition for* $R^{(0)}(\cdot)$ *in* (1.108) :

$$\inf_{x\in\mathbb{R}^n\setminus\mathcal{Z}}R^{(0)}(x)\geq 2\,; \tag{1.117}$$

Then for the weak solution $X(\cdot)$ *to* (1.103) *starting at any* $x_0 \in \mathbb{R}^n \setminus \mathcal{Z}$ *, we have*

$$\mathbb{Q}_{x_0}\big(X_1(t) = X_2(t) = X_3(t)\,, \quad \text{for some } t \geq 0\big) = 0\,.$$

Reasoning as in (1.101)-(1.102) *for the weak solution* $X(\cdot)$ *to* (1.13) *starting at* $x_0 \in \mathbb{R}^n \setminus \mathcal{Z}$ *, we obtain*

$$\mathbb{P}_{x_0}\big(X_1(t) = X_2(t) = X_3(t)\,, \quad \textit{for some } t \geq 0\big) = 0\,. \tag{1.118}$$

A class of examples satisfying (1.117) is given in Remarks 1.5-1.6.

On the other hand, regarding the presence of triple collisions we have the following.

Proposition 1.7. *Suppose that the matrices* $\sigma_\nu(\cdot)$, $\nu = 1, \cdots, m$ *in* (1.98) *are uniformly bounded and positive-definite, and*

$$\sup_{x \in \mathbb{R}^n \setminus \mathcal{Z}} R^{(0)}(x) < 2\,. \tag{1.119}$$

Then the weak solution $X(\cdot)$ *to* (1.103) *starting at any* $x_0 \in \mathbb{R}^n$ *satisfies*

$$\mathbb{Q}_{x_0}\big(X_1(t) = X_2(t) = X_3(t), \quad \text{for some } t \geq 0\big) \;=\; 1\,.$$

Moreover, if $\sup_{x \in \mathbb{R}^n \setminus \mathcal{Z}} R(x) < 2$ *where*

$$R(x) := \frac{\big[\mathrm{trace}(D'A(x)D) + 2\,x'DD'\mu(x)\big] \cdot x'DD'x}{x'DD'A(x)DD'x}\,; \quad x \in \mathbb{R}^n \setminus \mathcal{Z} \tag{1.120}$$

is a modification of $R^{(0)}(\cdot)$ *in* (1.108)*, then*

$$\mathbb{P}_{x_0}\big(X_1(t) = X_2(t) = X_3(t), \quad \text{for some } t \geq 0\big) \;=\; 1\,, \tag{1.121}$$

and we have an estimate similar to (1.113) :

$$\mathbb{P}_{x_0}\big(\, X_1(t) = X_2(t) = X_3(t), \quad \text{for some } t \in [0, T]\big) \geq 1 - \kappa(T\,; s(x_0), \delta_0) \tag{1.122}$$

where the distance function $s(\cdot)$ *and the tail probability* $\kappa(\cdot\,; \cdot, \cdot)$ *are given by* (1.99), (1.106) *and* (1.114) *now with dimension* $\delta_0 := \sup_{x \in \mathbb{R}^n \setminus \mathcal{Z}} R(x) < 2$ *, respectively.*

Proposition 1 and the first half of Proposition 2 are direct consequences of Lemma 1.12 and Section 1.6.2. Under the original probability measure $\mathbb{P}_{x_0}$, the process $s(X(\cdot))$ is a semimartingale with decomposition $d\,s(X(t)) = h(X(t))\,dt + d\,\Theta(t)$ where $h(\cdot)$ and $\Theta(\cdot)$ are obtained from $h^{(0)}(\cdot)$ and $\widetilde{\Theta}(\cdot)$ in (1.109), by replacing $R^{(0)}(\cdot)$ in (1.108) by $R(\cdot)$ in (1.120) and $\widetilde{W}(\cdot)$ in (1.18) by $W(\cdot)$. Then, the comparison with Bessel processes is repeated in the similar manner. Thus, if $\sup_{x\in\mathbb{R}^n\setminus\mathcal{Z}} R(x) < 2$, we obtain (1.121) and the estimate (1.122).

Remark 1.4. Since $A(\cdot)$ is positive-definite and $\text{rank}(D) = 2$, the matrix $D'A(\cdot)D$ is non-negative-definite and the number of its non-zero eigenvalues is equal to $\text{rank}(D'A(\cdot)D) = 2$. This implies

$$R^{(0)}(x) \geq \frac{\sum_{i=1}^{3} \lambda_i^D(x)}{\max_{1\leq i\leq 3} \lambda_i^D(x)} > 1; \quad x \in \mathbb{R}^n \setminus \mathcal{Z}\,,$$

where $\{\lambda_i^D(\cdot), i = 1,2,3\}$ are the eigenvalues of the (3×3) matrix $D'A(\cdot)D$.

On the other hand, an upper bound for $R^{(0)}(\cdot)$ is given by

$$R^{(0)}(x) \leq \frac{\text{trace }(D'A(x)D)}{3\min_{1\leq i\leq n}\lambda_i(x)}; \quad x \in \mathbb{R}^n \setminus \mathcal{Z}\,, \tag{1.123}$$

where $\{\lambda_i(\cdot), 1 \leq i \leq n\}$ are the eigenvalues of $A(\cdot)$. In fact, we can verify $DD'DD' = 3DD'$, $\{x \in \mathbb{R}^n : DD'x = 0\} = \mathcal{Z}$, and so if $DD'x \neq 0 \in \mathbb{R}^n$,

$$\min_{1\leq i\leq n} \lambda_i(x) \leq \frac{x'DD'A(x)DD'x}{x'DD'DD'x} = \frac{Q(x)}{3} = \frac{\text{trace }(D'A(x)D)}{3R^{(0)}(x)}\,;$$

this gives the upper bound (1.123) for $R^{(0)}(\cdot)$ above. $\square$

Remark 1.5. For the standard, $n-$dimensional Brownian motion, i.e., $\sigma(\cdot) \equiv I_n$, $n \geq 3$, the quantity $R^{(0)}(\cdot)$ of (1.108) is computed easily: $R^{(0)}(\cdot) \equiv 2$. More generally, suppose that the variance covariance rate $A(\cdot)$ is

$$A(x) := \sum_{\nu=1}^{m} (\alpha_\nu I_n + \beta_\nu DD' + \mathbb{I}\,\mathbb{I}'\ \text{diag}(\gamma_\nu)) \cdot \mathbf{1}_{\mathcal{R}_\nu}(x)\,; \quad x \in \mathbb{R}^n\,,$$

for some scalar constants α_ν, β_ν and $(n\times 1)$ constant vectors γ_ν, $\nu = 1,\ldots,m$. Here $\text{diag}(x)$ is the $(n\times n)$ diagonal matrix whose diagonal entries are the elements of $x \in \mathbb{R}^n$, and $\mathbb{I}$ is the $(n\times 1)$ vector with all entries equal to one. Then $R^{(0)}(\cdot) \equiv 2$ in $\mathbb{R}^n \setminus \mathcal{Z}$, because $\mathbb{I}'D = (0,0,0) \in \mathbb{R}^{1\times 3}$

and

$$DD' = \frac{1}{3} DD'DD' = \begin{pmatrix} 2 & -1 & -1 & \\ -1 & 2 & -1 & 0 \\ -1 & -1 & 2 & \\ & 0 & & 0 \end{pmatrix} \in \mathbb{R}^{n \times n}.$$

Hence, if the coëfficients α_ν, β_ν and γ_ν, $\nu = 1, \ldots, m$ are chosen above so that $A(\cdot)$ is positive-definite, then we have (1.118). $\square$

Remark 1.6. The condition (i) of (1.117) in Proposition 1 holds under several circumstances. For example, take $n = 3$ and fix the elements $a_{11}(\cdot) = a_{22}(\cdot) = a_{33}(\cdot) \equiv 1$ of the symmetric matrix $A(\cdot) = \sigma\sigma'(\cdot)$ in (1.105) and choose the other parameters by

$$\begin{aligned} a_{12}(x) = a_{21}(x) &:= \alpha_{1+} \mathbf{1}_{\mathcal{R}_{1+}}(x) + \alpha_{1-} \mathbf{1}_{\mathcal{R}_{1-}}(x)\,, \\ a_{23}(x) = a_{32}(x) &:= \alpha_{2+} \mathbf{1}_{\mathcal{R}_{2+}}(x) + \alpha_{2-} \mathbf{1}_{\mathcal{R}_{2-}}(x)\,, \\ a_{31}(x) = a_{13}(x) &:= \alpha_{3+} \mathbf{1}_{\mathcal{R}_{3+}}(x) + \alpha_{3-} \mathbf{1}_{\mathcal{R}_{3-}}(x)\,; \quad x \in \mathbb{R}^3\,, \end{aligned} \tag{1.124}$$

where $\mathcal{R}_{i\pm}$, $i = 1, 2, 3$ are subsets of $\mathbb{R}^3$ defined by

$$\begin{aligned} &\mathcal{R}_{1+} := \{\, x \in \mathbb{R}^3 \,:\, \mathfrak{f}_1(x) > 0 \,\}\,, \quad \mathcal{R}_{2+} := \{\, x \in \mathbb{R}^3 \,:\, \mathfrak{f}_1(x) = 0\,, \mathfrak{f}_2(x) > 0 \,\}\,, \\ &\mathcal{R}_{1-} := \{\, x \in \mathbb{R}^3 \,:\, \mathfrak{f}_1(x) < 0 \,\}\,, \quad \mathcal{R}_{2-} := \{\, x \in \mathbb{R}^3 \,:\, \mathfrak{f}_1(x) = 0\,, \mathfrak{f}_2(x) < 0 \,\}\,, \\ &\mathcal{R}_{3+} := \{\, x \in \mathbb{R}^3 \,:\, \mathfrak{f}_1(x) = \mathfrak{f}_2(x) = 0\,, \mathfrak{f}_3(x) > 0 \,\}\,, \\ &\mathcal{R}_{3-} := \{\, x \in \mathbb{R}^3 \,:\, \mathfrak{f}_1(x) = \mathfrak{f}_2(x) = 0\,, \mathfrak{f}_3(x) < 0 \,\}\,, \\ &\mathfrak{f}_1(x) := [x_3 - x_1 - (-2 + \sqrt{3})(x_2 - x_3)] \cdot [x_3 - x_1 - (-2 - \sqrt{3})(x_2 - x_3)]\,, \\ &\mathfrak{f}_2(x) := [x_2 - x_3 - (-2 + \sqrt{3})(x_1 - x_2)] \cdot [x_2 - x_3 - (-2 - \sqrt{3})(x_1 - x_2)]\,, \\ &\mathfrak{f}_3(x) := [x_1 - x_2 - (-2 + \sqrt{3})(x_3 - x_1)] \cdot [x_1 - x_2 - (-2 - \sqrt{3})(x_3 - x_1)]\,, \end{aligned}$$

for $x \in \mathbb{R}^3$ with the six constants $\alpha_{i\pm}$ satisfying $0 < \alpha_{i+} \le 1/2$, $-1/2 \le \alpha_{i-} < 0$ for $i = 1, 2, 3$. Note that the zero set $\mathcal{Z}$ defined in (1.107) is $\{\, x \in \mathbb{R}^3 \,:\, \mathfrak{f}_1(x) = \mathfrak{f}_2(x) = \mathfrak{f}_3(x) = 0 \,\}$. Thus, we split the region $\mathbb{R}^3 \setminus \mathcal{Z}$ into six disjoint polyhedral regions $\mathcal{R}_{i\pm}$, $i = 1, 2, 3$.

Remark 1.7. In the example of Bass & Pardoux [9], mentioned briefly in the Introduction, the diffusion matrix $\sigma(\cdot) = \sum_{\nu=1}^{m} \sigma_\nu(\cdot) \mathbf{1}_{\mathbb{R}_\nu}(\cdot)$ in (1.98) has a special characteristic in the allocation of its eigenvalues: *All eigenvalues but the largest are small*, namely, they are of the form $(1, \varepsilon, \cdots, \varepsilon)$, where $0 < \varepsilon < 1/2$ satisfies, for some $0 < \delta < 1/2$:

$$\left| \frac{x'\sigma(x)\sigma(x)'x}{\|x\|^2} - 1 \right| \leq \delta \qquad \text{for } x \in \mathbb{R}^n, \quad \text{and} \quad \frac{(n-1)\varepsilon^2 + \delta}{1-\delta} < 1\,. \tag{1.125}$$

This is the case when the diffusion matrix $\sigma(\cdot)$ can be written as a piecewise constant function $\sum_{\nu=1}^{m} \sigma_\nu \mathbf{1}_{\mathcal{R}_\nu}(\cdot)$, where the constant $(n \times n)$ matrices $\{\sigma_\nu,\ \nu = 1 \ldots m\}$ have the decomposition

$$\sigma_\nu \sigma_\nu' := (y_\nu\,,\, B_\nu)\ \mathrm{diag}\,(1, \varepsilon^2, \ldots, \varepsilon^2) \begin{pmatrix} y_\nu' \\ B_\nu' \end{pmatrix},$$

the fixed $(n \times 1)$ vector $y_\nu \in \mathbb{R}_\nu$ satisfies

$$\|y_\nu\| = 1\,, \qquad \frac{|\langle\, x\,,\, y_\nu\,\rangle|^2}{\|x\|^2} \geq 1 - \varepsilon\,; \quad x \in \mathbb{R}_\nu\,, \tag{1.126}$$

and the $(n \times (n-1))$ matrix B_ν consists of $(n-1)$ orthonormal $n-$dimensional vectors orthogonal to each other and orthogonal to y_ν, for $\nu = 1, \ldots m$. Then, for all $x \in \mathbb{R}^n$, we have

$$\frac{\|x\|^2 \mathrm{trace}\,(\sigma(x)\sigma(x)')}{x'\sigma(x)\sigma(x)'x} - 1 \leq \frac{(n-1)\varepsilon^2 + \delta}{1-\delta} < 1.$$

This is sufficient for the process $X(\cdot)$ to hit the origin in finite time.

To exclude this situation, we introduce the *effective dimension* $\mathrm{ED}_{\mathcal{A}}(\cdot)$ of the elliptic second-order operator $\mathcal{A}$ defined in (1.104), namely

$$\mathrm{ED}_{\mathcal{A}}(x) := \frac{\|x\|^2 \mathrm{trace}\,(\sigma(x)\sigma(x)')}{x'\sigma(x)\sigma(x)'x} = \frac{\|x\|^2 \mathrm{trace}\,(A(x))}{x'A(x)x} \tag{1.127}$$

for $x \in \mathbb{R}^n \setminus \{0\}$. This function comes from the theory of the so-called *exterior Dirichlet problem* for second-order elliptic partial differential equations, pioneered by Meyers & Serrin [44]. These authors show that

$$\inf_{x \in \mathbb{R}^n \setminus \{0\}} \mathrm{ED}_{\mathcal{A}}(\cdot) > 2 \tag{1.128}$$

is a sufficient criterion for the existence of solution to an exterior Dirichlet problem. In a manner similar to the proof of Proposition 1, it is possible to show that (1.128) is sufficient for $\mathbb{P}_{x_0}(X_1(t) = \cdots = X_n(t) = 0 \text{ for some } t \geq 0) = 0$, since $R^{(0)}(\cdot)$ becomes $\mathrm{ED}_{\mathcal{A}}(\cdot)$ when the matrix D is replaced by the identity matrix.

With $\sigma(\cdot)$ as in (1.98), the effective dimension $\mathrm{ED}_{\mathcal{A}}(\cdot)$ satisfies

$$\mathrm{ED}_{\mathcal{A}}(x) \geq \min_{\nu=1,\cdots,m} \left(\frac{\|x\|^2 \text{ trace } (\sigma_\nu(\cdot)\sigma'_\nu(\cdot))}{x'\sigma_\nu(\cdot)\sigma'_\nu(\cdot)x} \right) \geq \min_{\nu=1,\cdots,m} \left(\frac{\sum_{i=1}^n \lambda_{i\nu}(x)}{\max_{i=1,\cdots,n} \lambda_{i\nu}(x)} \right)$$

for $x \in \mathbb{R}^n \setminus \{0\}$, where $\{\lambda_{i\nu}(\cdot),\, i = 1, \cdots, n\}$ are the eigenvalues of the matrix-valued functions $\sigma_\nu(\cdot)\sigma'_\nu(\cdot)$, for $\nu = 1, \cdots, m$. Thus, $\mathrm{ED}_{\mathcal{A}}(\cdot) > 2$ if

$$\inf_{x \in \mathbb{R}^n \setminus \{0\}} \min_{\nu=1,\cdots,m} \left(\frac{\sum_{i=1}^n \lambda_{i\nu}(x)}{\max_{i=1,\cdots,n} \lambda_{i\nu}(x)} \right) > 2\,,$$

a condition that can be interpreted as mandating that the relative size of the maximum eigenvalue is not too large, when compared to the other eigenvalues.

Note that in the example of Bass & Pardoux [9], *all n Brownian particles collide at the origin at the same time, infinitely often:* for $x_0 \in \mathbb{R}^n$ we have

$$\mathbb{Q}_{x_0}\big(X_1(t) = X_2(t) = \cdots = X_n(t) = 0\,, \quad \text{for infinitely many } t > 0\big) = 1\,.$$

This is a special case of Proposition 2. Under the setting (1.125) it is seen that $R^{(0)}(\cdot) \leq 2 - \eta$ for some $\eta > 0$. In fact, it follows from (1.126) that there exists a constant $\xi \in (0, 1/2)$ such that

$$\frac{|\langle\, D'x\,, D'y_\nu \,\rangle|^2}{\|D'x\|^2 \cdot \|D'y_\nu\|^2} \geq 1 - \xi\,; \quad x\,, y_\nu \in \mathbb{R}^n \setminus \mathcal{Z}\,,$$

and we obtain

$$R^{(0)}(x) = \sum_{\nu=1}^m \frac{(1-\varepsilon^2)\|D'y_\nu\|^2 + 6\varepsilon^2}{(1-\varepsilon^2)\frac{|\langle\, D'x\,, D'y_\nu\,\rangle|^2}{\|D'x\|^2} + 3\varepsilon^2} 1_{\mathcal{R}_\nu}(x) \leq \frac{1}{1-\xi} < 2\,; \quad x \in \mathbb{R}^n \setminus \mathcal{Z}\,.$$

$\square$

Remark 1.8. Friedman [13] established theorems on the non-attainability of lower-dimensional manifolds by non-degenerate diffusions. Let $\mathcal{M}$ be a closed $k-$dimensional C^2-manifold in $\mathbb{R}^n$, with $k \leq n-1$. At each point $x \in \mathcal{M}$, let $N_{k+i}(x)$ form a set of linearly independent vectors in $\mathbb{R}^n$ which are *normal to* $\mathcal{M}$ and x. Consider the $(n-k) \times (n-k)$ matrix $\alpha(x) := (\alpha_{ij}(x))$ where

$$\alpha_{ij}(x) = \langle A(x) N_{k+i}(x), N_{k+j}(x) \rangle\,; \quad 1 \leq i, j \leq n-k\,,\ x \in \mathcal{M}\,.$$

Roughly speaking, the strong solution of (1.13) under linear growth condition and Lipschitz condition on the coëfficients cannot attain $\mathcal{M}$, if rank $(\alpha(x)) \geq 2$ holds for all $x \in \mathcal{M}$. The rank indicates how wide the orthogonal complement of $\mathcal{M}$ is. If the rank is large, the manifold $\mathcal{M}$ is too *thin* to be attained. The fundamental lemma there is based on the solution $u(\cdot)$ of partial differential inequality $\mathcal{A}u(\cdot) \leq \mu u(\cdot)$ for some $\mu \geq 0$, outside but near $\mathcal{M}$ with $\lim_{\text{dist}(x,\mathcal{M})\to\infty} u(x) = \infty$, which is different from our treatment in the previous sections.

Ramasubramanian [48] [49] examined the recurrence and transience of projections of weak solution to (1.13) for *continuous* diffusion coëfficient $\sigma(\cdot)$, showing that any $(n-2)-$dimensional C^2-manifold is not hit. The integral test developed there has an integrand similar to the effective dimension studied in [44], as pointed out by M. Cranston in MathSciNet Mathematical Reviews on the Web.

The above Propositions 1 and 2 are complementary with those previous general results, since the coëfficients here are allowed to be *piece-wise continuous*, however, they depend on the typical geometric characteristic on the manifold $\mathcal{Z}$ we are interested in. Since the manifold of interest in this work is the zero set $\mathcal{Z}$ of the function $s(\cdot)$, the projection $s(X(\cdot))$ of the process and the corresponding effective dimensions $\text{ED}_{\mathcal{A}}(\cdot)$, $R^{(0)}(\cdot)$ are studied. $\square$

Remark 1.9. As V. Papathanakos first pointed out, the conditions (1.117), (1.119) in Propositions 1 and 2 are disjoint, and there is a "gray" zone of sets of coëfficients which satisfy neither of the conditions. This is because we compare with Bessel processes, replacing the n-dimensional problem by a solvable *one*-dimensional problem. In order to look at a finer structure, we discuss a special case in the next section by reducing it to a *two*-dimensional problem. This follows a suggestion of A. Banner. $\square$

Chapter 2

Brownian motions with reflection

In this chapter our goal is to study attainability and ergodic behavior of multidimensional diffusion $Y(\cdot)$ in a sub-domain $\mathfrak{S} \subset \mathbb{R}^{n-1}$, where $Y(\cdot)$ is understood as a transformation $Y(\cdot) := F(X(\cdot))$ of the $n-$dimensional process $X(\cdot)$ in Chapter 1. Typically, we consider the case of $\mathfrak{S} := (\mathbb{R}_+)^{n-1}$, i.e., that all coördinates are positive, so that the transformed process $F(X(\cdot))$ stays in the non-smooth domain $(\mathbb{R}_+)^{n-1}$. For example, the mapping $\mathcal{P}(\cdot)$ defined in (1.3) in Section 1.1.1 in Chapter 1 does this job.

One of interesting questions is the behavior of $Y(\cdot) := F(X(\cdot))$ near the non-smooth boundary of $\mathfrak{S}$. For the domains with smooth boundary Stroock & Varadhan [54] introduced submartingale problem to formulate existence and uniqueness of process. The non-smooth boundary of $(\mathbb{R}_+)^{n-1}$ introduces some interesting technicality, in addition to their analysis, and affects the whole study of the process.

We start with the process $Y(\cdot)$ restricted in one-dimensional half space $\mathbb{R}_+$, i.e., $n = 2$ in Section 2.1. There it is natural to examine the local time of $Y(\cdot)$ at the origin which is a random measure of how much the process stays around the origin. The local time has good properties, and appears as an ingredient of the submartingale problem. Then, we analyze the process restricted in two-dimensional polyhedron $(\mathbb{R}_+)^2$, i.e., $n = 3$, and we generalize to the multidimensional case in Section 2.3. In higher dimension the random measure of how much the process stays around the boundary is not the good one-dimensional local time anymore. A key component in understanding the process $Y(\cdot) = F(X(\cdot))$ is attainability for some non-smooth domains. We discuss attainability of non-smooth part for the study of uniqueness of process in each case.

Another interesting question is ergodicity; it will be discussed after Section 2.4. We consider the stationary distribution of the multidimensional Brownian motion with reflection. The stationary distribution can be written as the product of exponential distribution functions, under the skew-symmetry condition (2.60) on the diffusion coëfficients. The so-called basic adjoint relation (2.109) on differential operators (2.95) and their adjoints (2.96) play an essential role to determine the stationary distribution. We will see that the skew-symmetry condition, appearing in Proposition 2.4 and Proposition 2.6, have deep connection with attainability of non-smooth part of $\mathfrak{S}$. The study of ergodic behavior of $F(X(\cdot))$ leads us to the understanding of ergodic property of the original process $X(\cdot)$ in Chapter 3.

2.1 One-dimensional Positive Half Line

In this section we construct the Brownian motion on the half-line $[0, \infty)$, which we call the one-dimensional reflected Brownian motion. By the symmetry of Brownian motion between $(0, \infty)$ and $(-\infty, 0)$ and by the strong Markov property, it is easily obtained by taking the absolute value $|\cdot|$ of the one-dimensional Brownian motion. This non-linear transform makes all complication behind the scene. For example, define $\mathcal{Z}_\omega := \{0 \le t < \infty \,:\, W(t\,, \omega) = 0\,\}$ be the *zero set* of the Brownian motion $W(\cdot)$. Then, with probability one, the zero set $\mathcal{Z}_\omega$ is closed, unbounded, has an accumulation point at $t = 0$, has no isolated point in $(0, \infty)$ and therefore, it is dense itself. Especially, the Brownian motion $W(\cdot)$ starting at zero cannot stay in the half-line for any interval $[0, \delta]$ for any $\delta > 0$, i.e.,

$$\mathbb{P}^0(W_t \ge 0\,,\, 0 \le \forall t \le \delta) = 0\,, \tag{2.1}$$

from Blumenthal's $0-1$ law. In fact,

$$\begin{aligned}\mathbb{P}^0(\cup_{\delta>0}\{W_t \ge 0 \text{ for all } t \in [0, \delta]\}) &= \lim_{\delta\downarrow 0} \mathbb{P}^0(W_t \ge 0 \text{ for all } t \in [0, \delta]) \\ &\le \lim_{\delta\downarrow 0} \mathbb{P}^0(W_\delta \ge 0) = \frac{1}{2} < 1\,.\end{aligned}$$

The set in the left-hand is in $\mathcal{F}_{0+}$ and by the $0-1$ law, its probability is 0. Hence, for any $\delta > 0$, (2.1) holds.

From these facts, it is absolutely non-trivial to discuss the sample paths behavior of reflected Brownian motion $|W(\cdot)|$ near its zero set $\mathcal{Z}$. In this section we see how we can handle the sample paths property of one-dimensional reflected Brownian motions.

Note that the reflected Brownian motion $|W(\cdot)|$ is a Markov process with transition density

$$p_+(t;x,y)dy := \mathbb{P}\big(|W(t+s)| \in dy \,\big|\, |W(s)| = x\,\big) = p(t;x,y) - p(t;x,-y)$$

where $p(t;x,y)$ is the transition probability density of Brownian motion, i.e.,

$$p(t;x,y) = \frac{1}{\sqrt{2\pi t}} e^{-\frac{(y-x)^2}{2t}}\,; \quad t > 0,\ (x,y) \in \mathbb{R}^2\,. \tag{2.2}$$

2.1.1 Skorokhod Equation

A.V. Skorohod [52] provided the path-wise construction of the reflected Brownian motion through the so-called Skorohod equation in the following way.

Let $z \geq 0$ be a real number, $y := \{y(t)\,:\,0 \leq t < \infty\}$ be a continuous function with $y(0) = 0$, $k := \{k(t)\,:\,0 \leq t < \infty\}$ be non-decreasing, continuous with $k(0) = 0$ and only increasing, when $\{\,t \geq 0\,:\,x(t) = 0\,\}$, i.e., $\int_0^t \mathbf{1}_{\{x(t)>0\}} d\,k(s) = 0$, which satisfy

$$x(t) := z + y(t) + k(t) \geq 0\,; \quad 0 \leq t < \infty\,. \tag{2.3}$$

Then, $k(\cdot)$ is given by

$$k(t) = \max\big[\,0,\ \max_{0\leq s\leq t}\{\,-(z+y(s))\,\}\,\big]\,; \quad 0 \leq t < \infty\,. \tag{2.4}$$

Now let $W := \{W_t\,:\,0 \leq t < \infty\}$ be the one-dimensional Brownian motion starting with $W_0 = z$ on some probability space $(\Omega\,,\mathcal{F}\,,\mathbb{P}^z)$.

Proposition 2.1 ([52])**.** *By this Skorohod equation, we introduce the mappings*

$$\begin{aligned}
&\Phi_t(z;\,y) := k(t) = \max\big[0,\ \max_{0\leq s\leq t}\{-(z+y(s))\}\big]\,; \quad 0 \leq t < \infty\,,\\
&\Psi_t(z;\,y) := x(t) = z + y(t) + \Phi_t(z;\,y)\,; \quad 0 \leq t < \infty\,,
\end{aligned} \tag{2.5}$$

and define a one-dimensional continuous process $X := \{X(t); 0 \leq t < \infty\}$ *by*

$$X(t) = \Psi_t(z;\,B.) = z - B_t + \Phi_t(z;\,B.)\,; \quad 0 \leq t < \infty\,, \tag{2.6}$$

where $B := \{B_\cdot\,; 0 \le t < \infty\}$ *is another Brownian motion on (possibly different) probability space* $(\Omega', \mathcal{F}', \mathbb{Q})$. *Then,* X *under the probability measure* $\mathbb{Q}$ *and* $|W| := \{|W(t)|\,; 0 \le t < \infty\}$ *under the probability measure* , $\mathbb{P}^z$ *have the same distribution.*

Thus, we obtain the probability distribution of reflected Brownian motion $|W(\cdot)|$ as that of $X(\cdot)$ through this Skorokhod equation from the Brownian motion from another Brownian motion $B(\cdot)$. Note that this construction is to obtain the process $X(\cdot)$ with the same probability distribution as $|W|$. The increasing part $\Phi_\cdot(z;\, B_\cdot)$ is identified as the local time, i.e.,

$$\Phi_t(z;\, B_\cdot) = k(t) = 2L_t(0)\,, \quad B_t = -\int_0^t \text{sgn}\,(W_s)\,dW_s \quad t \in \mathbb{R}^+\,,$$

from Tanaka's formula

$$|W_t| := |W_0| + \int_0^t \text{sgn}\,(W_s) dW_s + 2L_t(0)\,; \quad 0 \le t < \infty, \tag{2.7}$$

where $L_t(0)$ is the local time of W at zero. Here we define the local time of W at x, following P. Lévy, by

$$L_t(x) := \lim_{\varepsilon \downarrow 0} \frac{1}{4\varepsilon} \text{Leb}\,\{0 \le s \le t\,:\, |W_s - x| \le \varepsilon\,\}\,; \quad 0 \le t < \infty,\, x \in \mathbb{R}\,.$$

2.1.2 Submartingale Problem

In the previous subsection we saw that the Skorokhod equation gives one way of constructing the reflected Brownian motion. In this subsection we want to characterize every strong Markov families $\{\,\mathbb{P}^x\,\}$ of probability measures that have continuous paths, stay on the half line and behave like a standard Brownian motion away from 0 starting at x . This will be generalized and discussed precisely later in Section 2.3. Here we discuss formally and concisely.

Our starting point is again the fundamental characterization of Brownian martingales explained in Lemma 1.1 and the subsequent martingale Problem 1 in Chapter 1. Along with those characteristics of martingales, we restrict the process $X(\cdot)$ in the following way.

Problem 5. *We want to find a family of probability measures* $\{\,\mathbb{P}^x\,; x \in [0, \infty)\,\}$ *such that*

(1) *the canonical continuous process* $X(\cdot)$ *stays on the half line, i.e.,* $\mathbb{P}^x(X(t) \ge 0$ *for* $t \ge 0) = 1$, *starting at* $x \in \mathbb{R}_+$ *and has the semimartingale property with respect to some filtration* $\{\,\mathcal{F}_t\,; t \ge 0\}$ *satisfying the usual condition, and*

(2) *it behaves like a standard Brownian motion, i.e., the process*

$$M_t^f := f(X(t)) - f(X(0)) - \frac{1}{2}\int_0^t f''(X(s))d\,s\,; \quad 0 \le t < \infty\,, \tag{2.8}$$

is a continuous $\mathbb{P}^x-$*martingale for any* $f \in C^2(\mathbb{R})$ *with* $f'(0) = 0$.

Remark 2.1. First of all, applying $f(x) = x^2 \in \mathcal{C}^2(\mathbb{R})$ with $f'(0) = 0$ to the martingale property, we obtain that $X^2(t) - X^2(0) - t$ is a $\mathbb{P}^x-$martingale with zero expectation. This implies that the quadratic variation process satisfies $\langle X\rangle(t) = t$ for every $t \ge 0$. Since $\mathbb{P}^x(X(t) \ge 0) = 1$ from the above restriction (1) of Problem 5 on the process, we obtain

$$\mathbb{E}^x|\,X(t) - X(0)\,|^2 = \mathbb{E}^x[\,X^2(t) + X^2(0) - 2X(t)X(0)\,] \le 2x^2 + t < \infty\,. \tag{2.9}$$

Note that we added the restriction $f'(0) = 0$ on the underlying function $f(\cdot)$ in (2) of Problem 5. Then, the function $f(x) = x$ violates this restriction and is not applicable, which is different from the usual martingale problem discussed in Chapter 1. □

Remark 2.2. The following semimartingale property in Proposition 2.2, originally in S.R.S. Varadhan's online lecture note, indicates the well-posedness of Problem 5. For one-dimensional semimartingale $X(\cdot)$ it can be shown that there exists a local time process $L^X(\cdot)$ associated to $X(\cdot)$, however, for higher dimensional semimartingale the local time is not well-defined as the one-dimensional case. In this sense, we understand the following Proposition 2.2 is a special case that the semimartingale decomposition can be written in terms of the local time process. □

Proposition 2.2. *Under* $\mathbb{P}^x$, *the continuous semimartingale* X *constructed from the above* (1) *and* (2) *of Problem* 5 *is a non-negative submartingale with the decomposition*

$$X(t) = X(0) + M(t) + A(t)\,; \quad 0 \le t < \infty\,, \tag{2.10}$$

where $M(\cdot)$ *is a continuous martingale with* $M(0) = 0$ *and the continuous adapted non-decreasing process* $A(\cdot)$ *starting at* 0 *increases on the set* $\{t\,;\,X(t) = 0\,\}$.

Remark 2.3. For example, define

$$f_n(x) = \int_{-\infty}^x d\,y \int_{-\infty}^y n\,\rho(n\,z)\,d\,z\,; \quad x \in \mathbb{R}\,,\, n \ge 1\,, \tag{2.11}$$

where the function $\rho(\cdot)$ introduces the mollifier with $\rho(z) = c\cdot\exp[-1\,/\,\{\,(z-1)^2 - 1\,\}]\cdot\mathbf{1}_{\{0\le z\le 2\}}$

for $z \in \mathbb{R}$ and the constant c in this expression is chosen, so that $\int_{-\infty}^{\infty} \rho(z)\, d\, z = 1$. Then, the above function $f_n(\cdot)$ is in $C^2(\mathbb{R})$ with $f_n'(0) = 0$ satisfying

$$\lim_{n\to\infty} f_n(x) = x^+\,, \quad \lim_{n\to\infty} f_n'(x) = \mathbf{1}_{(0,\infty)}(x)\,, \quad f_n''(x) = n\,\rho(n\,x)\,; \quad x \in \mathbb{R}\,.$$

Applying the above martingale property to $f_n(\cdot)$, we obtain that

$$M_t^{f_n} := f_n(X(t)) - f_n(X(0)) - \int_0^t \frac{1}{2} f_n''(X(s)) d\, s\,; \quad 0 \le t < \infty$$

is a $\mathbb{P}^x-$martingale. Then, taking the limits on both sides with the first property (1) of non-negativity of process in the above Problem 5, we obtain the decomposition

$$X(t) = X(0) + M(t) + A(t)\,; \quad 0 \le t < \infty\,, \tag{2.12}$$

where $M(\cdot)$ is a limit of continuous martingale $\{\, M_{\cdot}^{f_n}\,\}$ and $A(\cdot)$ is the limit of the part of corresponding integrals. In fact, $M(\cdot)$ is the continuous $(\mathcal{F}_t)-$martingale and $A(\cdot)$ is the local time of X at level zero.

In fact, let $L(X) := \{\, L_t^a(X)\,;\, t \ge 0\,,\, a \ge 0\,\}$ be the family of local times $L_t^a(X)$ of the continuous semimartingale $X(\cdot)$ at level $a \in \mathbb{R}_+$ until time $t > 0$. Note that there is a modification of the local time which is continuous both in time and space variables. From the occupation-density formula, the non-negativity of X and right-continuity of the local time in space variable, it follows that

$$\lim_{n\to\infty} \int_0^t \frac{1}{2} f_n''(X(s))\, d\, s = \lim_{n\to\infty} \int_0^{\infty} \frac{1}{2} f_n''(x) L^x(t)\, d\, x = L^0(t)\,; \quad 0 \le t < \infty\,. \tag{2.13}$$

Moreover, again from non-negativity of X the limit $X(t) = X(t)^+ = \lim_{n\to\infty} f_n(X(t))$ exists $\mathbb{P}^x-$a.e. for $t \ge 0$ and so does the limit $M(t) := \lim_{n\to\infty} M_t^{f_n}$. Since the function $f_n(\cdot)$ defined in (2.11) satisfies $f_n(x) \le x$ for $x \ge 0$, $f_n''(\cdot) \ge 0$ and $\mathbb{E}(X^+(t)) \le (\mathbb{E}|X(t)|^2)^{1/2} < \infty$ from Remark 2.1, again by the dominated convergence theorem and the martingale property for $t \ge 0$ we obtain

$$\mathbb{E}^x[\, X(t)\,] = \lim_n \mathbb{E}^x[\, f_n(X(t))\,] = \lim_n \mathbb{E}^x\Big[f_n(X(0)) + \int_0^t \frac{1}{2} f_n''(X(s)) d\, s \Big] \ge \mathbb{E}^x[\, X(0)\,] = x\,.$$

That is, $X(\cdot)$ is a submartingale. Thus, the bound in (2.9) is, in fact,

$$\mathbb{E}^x|\, X(t) - X(0)\,|^2 = \mathbb{E}^x[\, X^2(t) + X^2(0) - 2X(t)X(0)\,] \leq 2x^2 + t - 2x^2 = t\,. \tag{2.14}$$

To show the limit $M(\cdot)$ is a continuous $\mathcal{F}_t-$martingale, we need some estimates on $\sup_n M_t^{f_n}$. We show in the next paragraph that

$$\mathbb{E}\Big[\sup_{n\geq 1} M_t^{f_n}\Big] < \infty\,; \quad 0 \leq t < \infty\,. \tag{2.15}$$

See (2.18) below.

In fact,

- we use the decomposition (2.10) and the following Gilat's theorem [15] saying that every non-negative submartingale is the absolute value of a martingale.

  Theorem 2.1 (Gilat[15]). *Let $X = \{\, X_t\,;\, t \geq 0\,\}$ be a non-negative submartingale with right continuous sample paths. Then, there exists a continuous martingale $Y = \{\, Y_t\,;\, t \geq 0\,\}$ such that the process $|Y| := \{\, |Y_t|\,;\, t \geq 0\,\}$ has the same distribution as X. Furthermore, the continuous martingale Y can be chosen either to be symmetric or else to have any mean smaller than $\mathbb{E}(X_0)$.*

  The process $X(\cdot)$ in (2.10) is a non-negative submartingale. From Gilat's theorem, there exists a martingale $Y(\cdot)$ such that the processes $X(\cdot)$ and $Y(\cdot)$ have the same probability distribution, and hence the couples $(X(\cdot)\,, \langle X\rangle(\cdot)\,)$ and $(|Y(\cdot)|\,, \langle Y\rangle(\cdot)\,)$ of processes have the same probability distribution. Thus, the local times $L_\cdot^a(X)$ of $X(\cdot)$ at level $a \geq 0$ has the same probability distribution with the local times $L_\cdot^a(|Y|)$ of $|Y(\cdot)|$ at level $a \geq 0$, since we can write

$$\begin{aligned} L_\cdot^a(X) &= \lim_{\varepsilon\downarrow 0} \varepsilon^{-1} \int_0^\cdot \mathbf{1}_{\{a\leq X(u)\leq a+\varepsilon\}}\, d\,X(u) \\ &\overset{(d)}{=} \lim_{\varepsilon\downarrow 0} \varepsilon^{-1} \int_0^\cdot \mathbf{1}_{\{a\leq |Y|(u)\leq a+\varepsilon\}}\, d\,\langle Y\rangle(u) = L_\cdot^a(|Y|) \qquad \text{in } \mathbb{R}_+\,, \end{aligned}$$

  where the second equality holds in the sense of distribution. Then, we estimate the supremum $\sup_{a\geq 0} L_t^a(X)$ of the local times $L_t^a(X)$ over the location parameter from the following theorem by Barlow and Yor [8].

  Theorem 2.2 (Barlow and Yor [8]). *Let M denote a continuous local $\mathcal{F}_t-$martingale and*

L be its bi-continuous family of local times, i.e.,

$$\lim_{b\downarrow a\,,s\downarrow t} L_s^b = L_t^a\,, \quad \lim_{b\uparrow a\,,s\uparrow t} L_s^b \text{ exists }.$$

Then, $L_t^* := \sup_{a\in\mathbb{R}} L_t^a$ is $\mathcal{F}_t-$adapted and left-continuous and satisfies a type of the Burkholder-Davis-Gundy inequality: there exist some universal constants $c>0$ and $C>0$ such that for any pair $(S\,,T)$ of stopping times with $S\le T$,

$$c\,\mathbb{E}\Big[\int_S^T d\,\langle M\rangle_u\Big]^{1/2} \le \mathbb{E}\Big[\sup_{a\in\mathbb{R}}\int_S^T d\,L_u^a\Big] \le C\,\mathbb{E}\Big[\int_S^T d\,\langle M\rangle_u\Big]^{1/2}.$$

Especially,

$$\mathbb{E}\Big[\sup_{a\in\mathbb{R}} L_t^a\Big] \le C\,\mathbb{E}\big[\,\langle M\rangle_t\,\big]^{1/2}\,; \quad t\ge 0\,. \tag{2.16}$$

Using this theorem, we obtain

$$\begin{aligned}
&\mathbb{E}\Big[\sup_{a\ge 0} L_t^a(X)\Big] = \mathbb{E}\Big[\sup_{a\ge 0} L_t^a(|Y|)\Big] = 2\mathbb{E}\Big[\sup_{a\ge 0} L_t^a(Y)\Big]\\
&\le 2\,C_1\,\mathbb{E}[\,\langle Y\rangle(t)\,]^{1/2} \quad (\text{ from (2.16)})\\
&\le C_2\,\mathbb{E}\Big[\sup_{0\le u\le t}|Y(u)|\Big] \quad (\text{the Burkholder-Davis-Gundy inequality})\\
&= C_2\mathbb{E}\Big[\sup_{0\le u\le t} X(u)\Big] \le 4\,C_2\,\mathbb{E}\big[X^2(t)\big] < \infty \quad (\text{Doob's submartingale inequality})\,.
\end{aligned}$$

It follows that

$$\begin{aligned}
&\mathbb{E}\Big[\sup_{n\ge 1}\int_0^t f_n''(X(u))d\,u\Big] = \mathbb{E}\Big[\sup_{n\ge 1}\int_0^t f_n''(z)L_t^a d\,a\Big]\\
&\le \mathbb{E}\Big[\sup_{n\ge 1}\int_0^\infty f_n''(a)d\,a\,\cdot\,\sup_{a\ge 0} L_t^a(X)\Big] = \mathbb{E}\Big[\sup_{a\ge 0} L_t^a(X)\Big] < \infty\,.
\end{aligned} \tag{2.17}$$

Thus, with $\mathbb{E}[\sup_{n\ge 1} f_n(X(t))] \le \mathbb{E}(X(t)) \le [\mathbb{E}(X^2(t))]^{1/2} < \infty$ we obtain

$$\mathbb{E}[\sup_{n\ge 1} M_t^{f_n}] \le \mathbb{E}(X(t)) + \mathbb{E}\Big[\sup_{n\ge 1}\int_0^t \frac{1}{2} f_n''(X(u))d\,u\Big] < \infty\,. \tag{2.18}$$

Therefore, the estimate (2.15) is now obtained.

- Recall that $f_\delta'(\cdot)$ is non-negative in $[0,\infty)$ with $f_\delta'(0)=0$ and is strictly positive in $(0,\infty)$.

Moreover, $\lim_{\delta\downarrow 0} f'_\delta(x) = \mathbf{1}_{\{x>0\}}$. From these facts, let us replace the condition (2.8) for functions $f(\cdot)$ in $C^2_o(\mathbb{R})$ with $f'(0)=0$ by the condition on functions in $C^2(\mathbb{R})$, namely that for every $C^2(\mathbb{R})$ function $g(\cdot)$ and for some non-decreasing continuous process $\widetilde{A}(\cdot)$,

$$(2.19) \qquad \widetilde{M}^g_t := g(X(t)) - g(X(0)) - \frac{1}{2}\int_0^t g''(X(s))d\,s - g'(0)\widetilde{A}(t); \quad 0 \le t < \infty$$

is a continuous $\mathbb{P}$-martingale.

Here we use this estimate and conclude by Doob's convergence theorem on the conditional expectations: for $s \ge t$,

$$\mathbb{E}[M(t)\,|\,\mathcal{F}_s] = \mathbb{E}\big[\lim_{n\to\infty} M^{f_n}_t\,|\,\mathcal{F}_s\big] = \lim_{n\to\infty}\mathbb{E}\big[M^{f_n}_t\,|\,\mathcal{F}_s\big] = \lim_{n\to\infty} M^{f_n}_s = M(s)\,.$$

Thus, $M(\cdot)$ is an $\mathcal{F}_t-$martingale. Therefore, $X(\cdot)$ has the decomposition (2.12). $\square$

Remark 2.4. Another example of such a function $f(\cdot)$, originally suggested in Varadhan's online lecture notes, is $f_\delta(x) = x - \delta\tan^{-1}(x/\delta)$ for $x\in\mathbb{R}$ and for fixed $\delta>0$. Since, $f'_\delta(x) = 1-\delta^2/(x^2+\delta^2)$, $f''_\delta(x) = 2\delta^2 x/(x^2+\delta^2)^2$, this function satisfies the above condition $f'(0)=0$. Moreover, $f''_\delta(x)>0$ for $x>0$. Then, $f_\delta(X(\cdot)) - f_\delta(X(0))$ is a sum of martingale $M^{f_\delta}_\cdot$ and non-decreasing bounded variation process $A^{f_\delta}_\cdot := \int_0^\cdot 2^{-1} f''_\delta(X(s))d\,s$, that is, a submartingale. Since $\int_0^\infty f''_\delta(x)d\,x = 1$, we use the same argument as in the previous example and obtain that the limit $X(\cdot) = \lim_{\delta\downarrow 0} f_\delta(X(\cdot))$ is a submartingale and it has the decomposition

$$X(t) = X(0) + M(t) + A(t)\,; \quad 0\le t<\infty\,,$$

where $M(\cdot) := \lim_{\delta\downarrow 0} M^{f_\delta}_\cdot$ and $A(\cdot) := \lim_{\delta\downarrow 0} A^{f_\delta}_\cdot = \lim_{\delta\downarrow 0}(1/2)\int_0^\cdot f''_\delta(X(s))d\,s = L^0_t(X)$.

2.2 Two-dimensional Wedge

Let $\mathfrak{S}$ be an infinite two-dimensional wedge with the angle $\xi\in(0,2\pi)$. Polar coördinates in $\mathbb{R}^2$ are denoted by (r,θ) for $r\ge 0$ and $\theta\in[0,2\pi]$. Then, the two-dimensional wedge $\mathfrak{S}$ can be written as

$$(2.20) \qquad \mathfrak{S} := \{(r,\theta)\,:\,0\le\theta\le\xi\,,\,r\ge 0\}$$

in the polar coördinates. When $\xi = \pi / 2$, it is called two-dimensional positive orthant, i.e., $\mathfrak{S} := \{x \in \mathbb{R}^2 \, : \, x_i \ge 0 \, , \, i = 1, 2\}$.

In this section the existence and uniqueness of a strong Markov process that has continuous sample paths on $\mathfrak{S}$ and has the following three properties are discussed:

1. the process X behaves like the standard two-dimensional Brownian motion in the interior $\overset{\circ}{\mathfrak{S}}$ of $\mathfrak{S}$ and the state space of the process is $\mathfrak{S}$,

2. the process reflects instantaneously at the boundary of the wedge with constant direction of reflection along each side $\partial\mathfrak{S}_1 := \{(r, \theta) \, : \, \theta = 0\}$ and $\partial\mathfrak{S}_2 := \{(r, \theta) \, : \, \theta = \xi\}$.

3. the amount of time that the process spends at the corner has Lebesgue measure zero.

These properties are intuitive. Mathematically the existence and uniqueness of such processes are formulated as submartingale Problem 6 below. This Brownian motion with reflection in $\mathfrak{S}$ is a special case of multidimensional diffusion with non-smooth boundary and singular boundary condition. It will be generalized into higher dimension in Section 2.3. The solution to this special submartingale problem is characterized by the boundary property explicitly, since there are some relations to solvable two-dimensional partial differential equations and special Green functions.

Set-up

Let θ_1 and θ_2 denote the angles that the directions of the reflections on the two sides $\partial\mathfrak{S}_1$ and $\partial\mathfrak{S}_2$ of the wedge make with the inward normals $\mathfrak{n}_1$ and $\mathfrak{n}_2$ to those sides, positive angles being toward the corner, respectively. The directions of reflection can be represented by vectors v_1 and v_2 with $n_i' v_i = \langle n_i \, , v_i \rangle = 1$ for $i = 1, 2$, respectively. See Figure 2.2.1. The scaler parameter $\beta := (\theta_1 + \theta_2) / \xi$ determines how much the process is pushed toward the corner by reflection at the boundary. This number β characterizes the process and its induced probability measure in the following submartingale problem.

Problem 6 (Submartingale Problem in two-dimensional wedge)**.** *Find a probability measure* $\mathbb{P}_x$ *on the space of continuous paths* $X(\cdot)$ *in* $\mathfrak{S}$ *such that*

1. $\mathbb{P}_x(X(0) = x) = 1$.

2. *For any twice continuously differentiable bounded function* $f(\cdot)$ *on* $\mathfrak{S}$ *which is constant in a neighborhood of the corner and has non-negative directional derivatives in the direction*

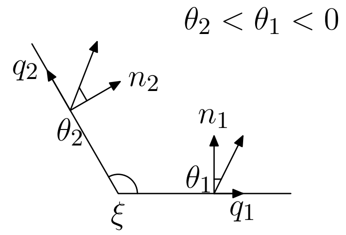

Figure 2.2.1: Two dimensional wedge.

of reflection, i.e., $D_i f \equiv \langle v_i, \nabla f \rangle \geq 0$ *for* $i = 1, 2$,

$$f(X(t)) - f(X(0)) - \frac{1}{2}\int_0^t \Delta f(X(s))d\,s\,; \quad 0 \leq t < \infty \tag{2.21}$$

is a $\mathbb{P}_x-$*submartingale. Here we define a differential operator* $\Delta := \sum_{i=1}^{2} \partial^2 / \partial\, x_i^2$.

3. *The process does not stay at the corner for the time of positive Lebesgue measure, i.e.,* $\mathbb{E}_x[\int_0^\infty \mathbf{1}_{\{0\}}(X(s))d\,s] = 0$.

Note that the uniqueness of solution of the submartingale problem starting from each $x \in \mathfrak{S}$ gives the strong Markov property.

Theorem 2.3 (Solution in the two dimensional wedge by Varadhan and Williams [59])**.** *The solution of the above Problem is characterized by* $\beta = (\theta_1 + \theta_2)\,/\xi$ *in the following.*

- *If* $\beta < 2$, *there is a unique solution* $\mathbb{P}_x$ *of the submartingale problem starting from* $x \in \mathfrak{S}$. *If* $x \neq 0$, *then the process* $X(\cdot)$ *reaches the corner of the wedge with* $\mathbb{P}_x-$*probability zero or one, respectively, according to whether* $\beta \leq 0$ *or* $0 < \beta < 2$.
- *If* $\beta \geq 2$, *then there is a unique* $\mathbb{P}_x$ *satisfying (1) and (2) but not (3) in the above Problem. It corresponds to the process starting at* x *which is almost surely absorbed at the corner.*

In the following subsections we sketch the proof of this theorem based on [59].

2.2.1 Existence

Absorbed process

First define $\tau(0) := \inf\{t > 0 \,:\, X(t) = 0\}$ for the the continuous functions $X(\cdot) \in C(\mathfrak{S})$ in $\mathfrak{S}$.

Lemma 2.1 ([59])**.** *For each* $x \in \mathfrak{S}$, *there exists a unique probability measure* $\mathbb{P}_x^0$ *such that*

1. $\mathbb{P}^0_x(X(0) = x) = 1$.

2. *For any twice continuously differentiable bounded function* $f(\cdot)$ *on* $\mathfrak{S}$ *which is constant in a neighborhood of the corner and has non-negative directional derivative in the direction of reflection, i.e. ,* $D_i f \equiv \langle v_i, \nabla f\rangle \geq 0$ *for* $i = 1, 2$ *, the process*

$$f(X(t \wedge \tau(0))) - f(X(0)) - \frac{1}{2}\int_0^{t\wedge\tau(0)} \Delta f(X(s))d\,s\,; \quad 0 \leq t < \infty \tag{2.22}$$

is a $\mathbb{P}^0_x-$*submartingale.*

3. $\mathbb{P}^0_x(X(t) = 0 \;\; \textit{for}\; t \geq \tau(0)\,) = 1$.

From the third property, such a process $X(\cdot)$ under $\mathbb{P}^0_x$ is called the absorbed process. It is understood by successive conditioning. A two-dimensioanl Brownian motion starting at x can be reflected obliquely from a single side, which is characterized by (2.22). By conditioning on the time of reflection from the single side, we successively construct the process. If the process undergoes an infinite number of successive reflections from the two sides, i.e., the process hits the corner, then the process is forced to stop. This step gives the last property. This is the intuitive interpretation of construction.

For the uniqueness, define a sequence $\{\,\sigma_n\,;\, n \geq 1\,\}$ of stopping times $\sigma_n := \inf\{t \geq 0 : \|X(t)\| = 1\,/\,n\,\}$. The uniqueness of the probability measure $\mathbb{P}^{\sigma_n}$ until σ_n is established by the theory of Stroock & Varadhan [54] for $n \geq 1$. This leads the uniqueness of $\mathbb{P}^0_x$ with the limiting procedure as $n \uparrow \infty$.

Let $D(\mathfrak{S})$ denote the space of all right continuous functions $X(\cdot)\,:\,[0,\infty) \mapsto \mathfrak{S}$ with finite left limits on $(0,\infty)$. We endow $D(\mathfrak{S})$ with the Skorokhod topology on $D(\mathfrak{S})$. The Borel $\sigma-$algebra $\mathcal{M}^D$ associated with this metric topology on $D(\mathfrak{S})$ is the same as that generated by the coördinate maps, so $\mathcal{M}^D = \sigma(X(t)\,:\,0 \leq t < \infty\,)$.

In order to solve submartingale Problem 6, we define an approximating sequence $\{\,\mathbb{P}^\delta\,;\,\delta \in \mathfrak{S}\setminus\{\,0\,\}\,\}$ of probability measures induced by the process $X(\cdot)$ with a jump at the corner; If $\tau(0)$ is finite, we move the process $X(\cdot)$ instantaneously to the points $\delta \in \mathfrak{S}\setminus\{\,0\,\}$ as soon as the process hits the corner. Then, the process is forced to start afresh at δ . This procedure is repeated forever.

The state space $\mathfrak{S}$ is locally compact and the trajectories have jumps of size at most $\|\delta\|$. Let $\eta > 0$ be fixed and take a small δ so that $\|\delta\| < 3\,\eta\,/\,4$. Pick $0 < r < \eta\,/\,4$ so that the balls

$B_x(r)$ with radius $r > 0$ and center at any $x \in \mathfrak{S} \cap \{y : \|y\| \geq \eta / 2\}$ interects at most one side of $\mathfrak{S}$. Define the first exit time $\tau_{x,\eta} := \inf\{t \geq 0 : X(t) \in B_x(\eta)\}$ from the ball $B_x(\eta)$ with radius $\eta > 0$ and center at x. Then, observe

$$\sup_{x \in \mathfrak{S}} \mathbb{P}_x^{\delta}(\tau_{x,\eta} \leq t) \leq \sup_{\|x\| \geq 3\eta / 4} \mathbb{P}_x^{\delta}(\tau_{x,r} \leq t) = \sup_{\|x\| \geq 3\eta / 4} \mathbb{P}_x^{0}(\tau_{x,r} \leq t) \xrightarrow[t \downarrow 0]{} 0 .$$

It follows that $\lim_{t\downarrow 0} \limsup_{\|\delta\|\downarrow 0} \sup_{x \in \mathfrak{S}} \mathbb{P}_x^{\delta}(\tau_{x,\eta} \leq t) = 0$ for each $\eta > 0$. Thus we obtain the following Lemma.

Lemma 2.2 (Lemma 2.3 of [59]). *The family $\{\mathbb{P}_x^{\delta}\}$ of probability measure is weakly relatively compact as $\|\delta\| \to 0$.*

Thus, by this Lemma 2.2 there exists a weak limit point $\mathbb{P}_x$ of $\{\mathbb{P}_x^{\delta_n} ; n \geq 1\}$, which will be a candidate of the solution to Problem 6. The first and second properties (1), (2) of Problem 6 are easily obtained by the limit procedure. In order to verify the third property (3), we want to estimate how much time the process spends in the neighborhood of the corner. In the next subsection we discuss if it reaches the corner or not, according to the magnitude of $\beta = (\theta_1 + \theta_2) / \xi$.

Attainability

In this subsection we categorize the processes into two cases (i) the process hits the corner $\beta > 0$ (ii) it never hits the corner $\beta \leq 0$.

First, we describe a $\mathbb{P}_x^0$−martingale $\Phi(X(\cdot))$ for some twice continuously differentiable function $\Phi(\cdot)$ on $\mathfrak{S} \setminus \{0\}$ satisfying the boundary value problem

$$\Delta \Phi = 0 \quad \text{in } \mathfrak{S} \setminus \{0\}, \quad D_i \Phi = 0 \quad \text{on } \partial\mathfrak{S}_i ; \ i = 1, 2 . \tag{2.23}$$

Then, by the submartingale characterization of $\mathbb{P}_x^0$ for the absorbed process and by Doob's stopping theorem, we obtain

$$\mathbb{E}_x^0[\Phi(X(t \wedge \tau_\varepsilon \wedge \tau_K))] = \Phi(x) ; \qquad x \in \{y : \varepsilon < \|y\| < K\} , \tag{2.24}$$

where

$$\tau_r := \inf\{t : \|X(t)\| = r\} ; \quad r \geq 0 . \tag{2.25}$$

Using this observation, we compute the probability $\mathbb{P}_x^0(\tau(0) < \infty)$ in polar coördinates.

Lemma 2.3 (Theorem 2.2 of [59])**.** *If* $\beta > 0$ *, then* $\mathbb{P}_x^0(\tau(0) < \infty) = 1$ *, while if* $\beta \le 0$ *, then* $\mathbb{P}_x^0(\tau(0) < \infty) = 0$ *for* $x \in \mathfrak{S} \setminus \{ 0 \}$ *.*

Proof. Solving the boundary problem (2.23) by separating variables, we obtain

$$\Phi(r, \theta) = \begin{cases} r^{\beta} \cos(\beta\theta - \theta_1) & \text{if } \beta \neq 0 \,, \\ \log r + \theta \tan \theta_1 & \text{if } \beta = 0 \,. \end{cases} \tag{2.26}$$

In fact, for example, if $\beta \neq 0$,

$$\nabla \Phi = (\partial \Phi \,/\, \partial r \,, (1 \,/\, r)\, \partial \Phi \,/\, \partial \theta) = (\beta r^{\beta - 1} \cos(\beta\theta - \theta_1)\,, -\beta r^{\beta - 1} \sin(\beta\theta - \theta_1)\,)$$

in polar coördinates. Since $v_1 = (\, -\tan\theta_1 \,, 1\,)$ and $v_2 = (\, -\tan\theta_2 \,, 1\,)$ in polar coördinates, $D_1 \Phi = v_1' \nabla \Phi = 0$ on $\{\, \theta = 0 \,, r > 0 \,\} = \partial \mathfrak{S}_1 \setminus \{\, 0 \,\}$ and $D_2 \Phi = v_2' \nabla \Phi = 0$ on $\{\, \theta = \xi \,, r > 0 \,\} = \partial \mathfrak{S}_2 \setminus \{\, 0 \,\}$. Since the function $\Phi(\cdot)$ is a linear combination of the real and imaginary parts of z^{β} for $z \in \mathbb{C}$, it can be extended to a twice continuously differentiable harmonic function on some domain containing $\mathfrak{S} \setminus \{\, 0 \,\}$. By applying Itô's formula to $\Phi(X(\cdot))$ and submartingale characterization, we verify that $\Phi(X(\cdot))$ is the local martingale. Since the process X has the same law as the process constructed path-wise from Skorokhod problem, it can be verified that τ_{ε} and τ_K is finite with probability one. Thus, $\Phi(X_{t \wedge \tau_{\varepsilon} \wedge \tau K})$ satisfies

$$\mathbb{E}_x^0[\Phi(X(t \wedge \tau_{\varepsilon} \wedge \tau_K))] = \Phi(x) \,; \quad x \in \{y \,:\, \varepsilon < \|y\| < K\,\} \,, \tag{2.27}$$

Now let p denote $\mathbb{P}_x^0(\tau_{\varepsilon} < \tau_K)$.
(i) If $\beta \neq 0$, it follows from (2.27) that

$$\varepsilon^{\beta} p + K^{\varepsilon}(1 - p)c \le \|x\|^{\beta} \,,$$

where $c := \min_{\theta \in [0, \xi]} \cos(\beta\theta - \theta_1) \ge \cos(|\theta_1| \vee |\theta_2|) > 0$. Then,

$$p \ge \frac{K^{\beta} - \|x\|^{\beta}/c}{K^{\beta} - \varepsilon^{\beta}} \quad \text{if } \beta > 0 \,, \quad p \le \frac{\|x\|^{\beta}/c - K^{\beta}}{\varepsilon^{\beta} - K^{\beta}} \quad \text{if } \beta < 0 \,,$$

and by letting $\varepsilon \downarrow 0$ first, then $K \uparrow \infty$, we obtain the desired result.

(ii) If $\beta = 0$, then it follows from (2.27) that

$$p \log \varepsilon + (1-p) \log K + \xi\, |\tan\theta_1| \geq \log\|x\| - \xi|\tan\theta_1|\,,$$

or

$$p \leq \frac{\log(K\,/\,\|x\|) + 2\xi\,|\tan\theta_1|}{\log(K/\varepsilon)}\,.$$

Therefore, by letting $\varepsilon \downarrow 0$ first, then $K \uparrow \infty$, we obtain the desired result. $\square$

Existence for $\beta < 2$

Now with these understandings we estimate how much time the process spends in the neighborhood of the corner. Define a twice continuous differentiable function $\Psi(\cdot)$ from the above $\Phi(\cdot)$ by

$$\Psi(r,\theta) = \begin{cases} \Phi(r,\theta) = r^{\beta}\cos(\beta\theta - \theta_1) & \text{if } \beta > 0\,, \\ e^{\Phi(r,\theta)} = r e^{\theta\tan\theta_1} & \text{if } \beta = 0\,, \\ 1\,/\,\Phi(r,\theta) = 1\,/\,(r^{\beta}\cos(\beta\theta - \theta_1)) & \text{if } \beta < 0\,, \end{cases} \tag{2.28}$$

and $\Psi(0,0) = 0$ with the polar coördinates. The function $\Psi(\cdot)$ is continuous on $\mathfrak{S}$ and $\Psi(r,\theta)$ is increasing with r for fixed θ. Thus, it suffices to look at $\mathbb{E}_x[\int_0^{\tau_K} \mathbf{1}_{[0,\varepsilon)}(\Psi(X(s)))d\,s]$ for the condition (3) $\mathbb{E}_x[\int_0^{\infty} \mathbf{1}_{\{0\}}(X(s))d\,s] = 0$ of Problem 6. Here the expectation is under the probability measure which is the weak limit of $\{\,\mathbb{P}_x^{\delta_n}\,;\, n \geq 1\}$.

Lemma 2.4 (An estimate of occupation time around the origin [59])**.** *Suppose that* $\beta < 2$ *and* $\Psi(x) < K$. *The expected occupation time around the origin is evaluated as, if* $x \neq 0$,

$$\begin{aligned} &\mathbb{E}_x\left[\int_0^{\tau_K} \mathbf{1}_{[0,\varepsilon)}(\Phi(X(s)))d\,s\right] \\ &\leq c^{-1}\liminf_{n\to\infty}\left[g(\Phi(x)) + \frac{\mathbb{P}_x^0(\tau(0) < \tau_K)\cdot g(\Phi(\delta_n))}{\mathbb{P}_{\delta_n}^0(\tau_K < \tau(0))]^{-1}}\right], \end{aligned} \tag{2.29}$$

and if $x = 0$,

$$\mathbb{E}_x\left[\int_0^{\tau_K} \mathbf{1}_{[0,\varepsilon)}(\Phi(X(s)))d\,s\right] \leq c^{-1}\liminf_{n\to\infty} g(\Phi(\delta_n))\,[\,\mathbb{P}_{\delta_n}^0(\tau_K < \tau(0))\,]^{-1}\,, \tag{2.30}$$

where $g(\cdot)$ *is defined with parameter* $\varepsilon > 0$ *in* (2.32)-(2.33) *below and* $c := \inf_z h(z)$ *is defined* *by*

$$h(z) = \big(\cos(\beta\theta - \theta_1)\big)^{2/\beta - 2}\mathbf{1}_{\{\beta \neq 0\}} + (1 + \tan^2\theta_1)e^{2\theta\tan\theta_1}\mathbf{1}_{\{\beta = 0\}}\,. \tag{2.31}$$

If $0 < \beta < 2$, or $2 < \beta$, let us define the continuous function $g(\cdot)$ by

$$g(y) := \frac{\varepsilon^{2/\beta}}{\beta}(1 - y/K)\mathbf{1}_{\{y \geq \varepsilon\}} + \frac{y(2\varepsilon^{2/\beta}(1-\varepsilon/K)/\beta + \varepsilon/K) - y^{2/\beta - 1}}{2 - \beta}\mathbf{1}_{\{0 < y \leq \varepsilon\}}\,, \tag{2.32}$$

and if $\beta = 0, 2$, we define

$$\begin{aligned} g(y) &:= \frac{y}{2}(\log\varepsilon + (1 - \varepsilon/K) - \log y)\mathbf{1}_{\{0 < y \leq \varepsilon\}} + \frac{\varepsilon}{2}(1 - y/K)\mathbf{1}_{\{\varepsilon \leq y\}}\,; \quad \beta = 2\,, \\ g(y) &:= ((\varepsilon^2 - e^{2y})/2 + \varepsilon^2\log(K/\varepsilon))\mathbf{1}_{\{y < \log\varepsilon\}} + \varepsilon^2(\log K - y)\mathbf{1}_{\{y \geq \log\varepsilon\}}\,; \quad \beta = 0\,, \end{aligned} \tag{2.33}$$

and if $\beta < 0$, we define

$$g(y) := -\frac{2\varepsilon^{1-2/\beta}(y - 1/K)}{\beta(2-\beta)}\mathbf{1}_{\{0 \leq y \leq 1/\varepsilon\}} + \left(\frac{\varepsilon^{-2/\beta} - y^{\beta/2}}{2-\beta} - \frac{2\varepsilon^{1-2/\beta}(1/\varepsilon - K)}{\beta(2-\beta)}\right)\mathbf{1}_{\{y \geq \varepsilon^{-1}\}}\,.$$

Moreover, letting $\varepsilon \downarrow 0$, we obtain that if $\beta < 2$,

$$\lim_{\varepsilon \downarrow 0}\mathbb{E}_x\left[\int_0^{\tau_K}\mathbf{1}_{[0,\varepsilon)}(\Phi(X(s)))d\,s\right] = 0$$

and hence (3) $\mathbb{E}_x[\int_0^\infty \mathbf{1}_{\{0\}}(X(s))d\,s] = 0$ of Problem 6 holds. This completes the proof of existence of solution in Theorem 2.3 for $\beta < 2$.

Non-existence for $\beta \geq 2$

Let us denote a probability measure induced from the process $X(\cdot)$ which satisfies 1 and 2 of Problem 6 by $\widehat{\mathbb{P}}^{\tau(0)}$. Given a positive $C_b^2(\mathfrak{S})-$function $f_{\varepsilon,K}$, we obtain by Itô's formula and Doob's martingale stopping theorem,

$$\widehat{\mathbb{E}}^{\tau(0)}[f_{\varepsilon,K}(t \wedge \tau_K))] = f_{\varepsilon,K}(0) + \frac{1}{2}\widehat{\mathbb{E}}^{\tau(0)}\Big[\int_0^{t\wedge\tau_K}\Delta f_{\varepsilon,K}(X(s))\,d\,s\,; \quad 0 \leq t < \infty\,, \tag{2.34}$$

where $\tau_K := \inf\{t > 0 \,:\, \Psi(X(t)) \geq K\}$. If we can find such a positive function $f_{\varepsilon\,,K}(\geq 0)$ that in $\{x \in \mathfrak{S} \,:\, 0 < \Psi(x) < K\}$,

$$\begin{aligned} &\lim_{\varepsilon\downarrow 0} f_{\varepsilon\,,K} = \Psi \ \text{ unifromly} \\ &\lim_{\varepsilon\downarrow 0} \Delta\, f_{\varepsilon\,,K} = 0 \quad \text{and } |\Delta f_{\varepsilon\,,K}| < c \ \text{ for some constant c,} \end{aligned} \tag{2.35}$$

then, by the dominated convergence theorem the limits of both sides of (2.34) yield

$$\widehat{\mathbb{E}}^{\tau(0)}\big[\Psi(X(t\wedge\tau_K))\big] = \Psi(0) = 0\,; \quad t \geq 0\,.$$

Since $\Psi > 0$ on $\mathfrak{S}\setminus\{0\}$, it implies that $X(t\wedge\tau_K) = 0$ $\widehat{\mathbb{P}}^{\tau(0)} - a.s.$. Now letting $K \uparrow \infty$, we obtain

$$\widehat{\mathbb{P}}^{\tau(0)}[X(t) = 0] = 1\,;\, t \geq 0. \tag{2.36}$$

This violates (3) of Problem 6. Therefore, if we can find $f_{\varepsilon\,,K}$ which satisfies (2.35), we can conclude that there is no solution which satisfies (1), (2) and (3) of Problem 6 starting from $x \in \mathfrak{S}$. In fact, let $\kappa : [0,\infty) \mapsto [0,1]$ be a twice continuously differentiable function which takes values zero in $[0,1/2]$ and one in $[1,\infty)$. Define for $0 < \varepsilon < K$

$$f_{\varepsilon\,,K}(x) = \kappa(\varepsilon^{-1}\Psi(x))\,(\,1 - \kappa(\,(2K)^{-1}\,\Psi(x)\,)\,)\,\Psi(x)\,; \quad 0 \in \mathfrak{S}\,. \tag{2.37}$$

Then, $\Delta\, f_{\varepsilon\,,K} = (\varepsilon^{-2}\, g''\Psi + 2\varepsilon^{-1}\, g')\,|\nabla\Psi|^2(x)\,\mathbf{1}_{\{\varepsilon\,/\,2\leq\Psi(x)\leq\varepsilon\}} \downarrow 0$, as $\varepsilon \downarrow 0$, and $|\Delta f_{\varepsilon\,,K} \leq c$ for some constant c. Thus, there is no solution of submartingale Problem 6 if $\beta > 2$.

2.2.2 Uniqueness

The proof of uniqueness of submartingale Problem 6 requires elaborate work as in the proof of martingale problem in Chapter 1. Here let us oversee the techniques that we can use.

Definition 2.1. For $\lambda > 0$ define a stopped resolvent

$$R_\lambda h(x) := \mathbb{E}_x\Big[\int_0^{\tau_0} e^{-\lambda\, t}\, h(X(t))\, d\,t\Big]\,; \quad x \in \mathfrak{S}\,,\, h \in C_b(\mathfrak{S})\,, \tag{2.38}$$

where τ_0 is defined through (2.25).

The uniqueness of submartingale Problem 6 can be shown through uniqueness of this stopped resolvent. Using a Green's function discussed in two dimensional wedge, we can represent the stopped resolvent in the following Lemma 2.5.

Lemma 2.5 ([59]). *The stopped resolvent* $R_\lambda(\cdot)$ *is equal to*

$$f(x) := 2\int_0^\xi \int_0^\infty G(\sqrt{2\,\lambda}r\,,\theta\,,\sqrt{2\,\lambda}s\,,t)\,h(s,t)\,s\,d\,s\,d\,t\,;\quad x\in\mathfrak{S}\,, \tag{2.39}$$

where $G(\cdot\,,\cdot\,,\cdot\,,\cdot)$ *is a Green's function for the operator* $\Delta - 1$ *in* $\overset{\circ}{\mathfrak{S}}$ *with boundary condition* $D_i f \geq 0$ *on* $\partial\mathfrak{S}_i$ *for* $i = 1, 2$ *, namely, if* $\beta \leq 0$ *, then*

$$\begin{aligned}G(r\,,\theta\,,s\,,t) := \frac{1}{2\pi\xi}\int_\infty^\infty d\,u \int_\infty^\infty d\,v\,\Big[&\cos(r\,\sinh u)\,\sin(s\,\sinh v)\\ &\cdot\frac{\rho(u+i\theta)\,\sinh[\pi\xi^{-1}(s+it)]}{\rho(v+it)\,[\cosh(\pi\xi^{-1}(v+is))-\cosh(\pi\xi^{-1}(u+i\theta))]}\Big]\,,\end{aligned}$$

and if $\beta > 0$ *, then*

$$\begin{aligned}G(r\,,\theta\,,s\,,t) := \frac{1}{2\pi\xi}\int_\infty^\infty d\,u \int_\infty^\infty d\,v\,\Big[&\sin(r\,\sinh u)\,\cos(s\,\sinh v)\\ &\cdot\frac{\rho(u+i\theta)\,\sinh[\pi\xi^{-1}(u+i\theta)]}{\rho(v+it)\,[\cosh(\pi\xi^{-1}(u+i\,\theta))-\cosh(\pi\xi^{-1}(v+i\,t))]}\Big]\,,\\ \rho(z) := e(z-i\,\xi\,,-\theta_1)\,e(z\,,\theta_2)\,,&\\ e(z,\theta) := \prod_{\substack{m\,,n:\ odd\\ positive}} \Big[1+z^2\big(m\xi+\frac{1}{2}\pi\,n+\theta\big)^{-2}\Big]&\Big[1+z^2\big(m\xi+\frac{1}{2}\pi\,m+\theta\big)^{-2}\Big]^{-1}.\end{aligned}$$

Moreover, the above $f(\cdot)$ *in* (2.39) *is continuous at the origin and bounded.*

Sketch of proof. The behavior of $G(\,\cdot\,,\cdot\,,s\,,t\,)$ near $(\,s\,,t\,)$ is

$$G(\,r\,,\theta\,,s\,,t) = -\frac{1}{4\pi}\log[r^2+s^2-2\,r\,s\cos(\theta-t)]+O(1)\,, \tag{2.40}$$

as $|r-s|+|\theta-t| \to 0$. Using this property, one can show that the function $f(\cdot)$ in (2.39) satisfies

$$\begin{cases} \left(\frac{1}{2}\Delta-\lambda\right)f = -h\,,\\ D_i\,f = 0 \quad \text{in } \partial\mathfrak{S}_i\setminus\{0\}\,;\quad i=1,2\,, \text{ and}\\ \text{if } \beta>0\,,\quad f(0,\theta)=0\,. \end{cases} \tag{2.41}$$

By Itô's formula and Doob's stopping theorem,

(2.42)
$$f(x) = \mathbb{E}_x\big[e^{-\lambda(t\wedge\tau_0)} f(X(t\wedge\tau_0))\big] + \mathbb{E}_x\Big[\int_0^{t\wedge\tau_0} e^{-\lambda\, s}\, h(X(s))\, d\, s\,\Big]; \quad f \in C_b^2(\mathfrak{S}\setminus\{\Psi(\cdot)<\varepsilon\})\,.$$

It follows from Lemma 2.3 that if $\beta \le 0$, $\mathbb{P}_x[\tau_0 = \infty] = 1$, and if $\beta > 0$, we can verify $f(0) = 0$ from the property of Green's function. Thus, in either case, we obtain (2.38). $\square$

Uniqueness for $\beta \le 0$

Using Lemma 2.5, we show uniqueness for $\beta \le 0$. Let us define $\sigma_\varepsilon := \inf\{t \ge 0 : \Psi(X(t)) \ge \varepsilon\}$ for $\varepsilon > 0$ and denote by $\widehat{\mathbb{P}}^{\sigma_\varepsilon}$ the regular conditional probability distribution of $\mathbb{P}|\mathcal{M}_{\sigma_\varepsilon}$, restricted on $\mathcal{F}_{\sigma_\varepsilon}$. Since the amount of time that the process $X(\cdot)$ spends stay at the origin is Lebesgue measure zero, $h(\cdot)$ in (2.38) can be restricted in a class of functions which are zero in a neighborhood of the origin. There exists ε_0 such that for $0 < \varepsilon < \varepsilon_0$

$$f(X(\sigma_\varepsilon)) = \widehat{\mathbb{E}}^{\sigma_\varepsilon}\Big[\int_0^\infty e^{-\lambda\, t}\, h(X(t))\, d\, t\Big]. \tag{2.43}$$

Then, since $\Psi(\cdot) \ge 0$ and hence $\sigma_\varepsilon \downarrow 0$ as $\varepsilon \downarrow 0$, and moreover $f(\cdot)$ is continuous at the origin and bounded as is stated in Lemma 2.5, we obtain

$$\begin{aligned}
\mathbb{E}_x\Big[\int_0^\infty e^{-\lambda\, t} h(X(t))\, d\, t\Big] &= \mathbb{E}_x\Big[\Big(\int_0^{\sigma_\varepsilon} + \int_{\sigma_\varepsilon}^\infty\Big) e^{-\lambda\, t} h(X(t))\, d\, t\Big] \\
&= \mathbb{E}_x\Big[\int_0^{\sigma_\varepsilon} h(X(t))\, d\, t\Big] + \mathbb{E}_x\Big[e^{-\lambda\sigma_\varepsilon}\cdot \widehat{\mathbb{E}}^{\sigma_\varepsilon}\Big[\int_0^\infty e^{-\lambda\, t} h(X(t))\, d\, t\Big]\Big] \\
&= \lim_{\varepsilon\downarrow 0} \mathbb{E}_x\big[f(X(\sigma_\varepsilon)) e^{-\lambda\sigma_\varepsilon}\big] = f(0)\,; \qquad \text{for all } \mathbb{P}_x\,,\ x \in \mathfrak{S}\,.
\end{aligned}$$

This implies the uniqueness of solution $\mathbb{P}_x$ of the submartingale Problem 6.

Uniqueness for $0 < \beta < 2$

Since $\mathbb{P}_x(\tau_0 < \infty) = 1$ where τ_0 is defined through (2.25) in this case of $0 < \beta < 2$ from Lemma 2.3, it takes more work to identify the behavior around the origin. We leave most of computation related to Green's function and conclude Lemma 2.6.

Lemma 2.6 ([59]). *For $h \in C_b(\mathfrak{S})$ with $h(z) = O(e^{-c|z|})$ as $|z| \to \infty$ with some constant*

$c > 0$,

$$\lim_{r\downarrow 0}[\Psi(x)]^{-1}\mathbb{E}_x\Big[\int_0^{\tau(0)} h(X(t))\,d\,t\Big] = C(h)\,, \tag{2.44}$$

where $C(h)$ *is a strictly positive constant which depends on* h, *whenever* $h \ge 0$ *and* $h \not\equiv 0$, *given by*

$$C(h) = \frac{4\pi^2}{\theta_1+\theta_2}\int_0^{\xi}\int_0^{\infty} s^{1-\alpha}\,\cos(\beta t-\theta_1)h(s,t)\,d\,s\,d\,t\,. \tag{2.45}$$

and moreover,

$$\mathbb{E}_0\Big[\int_0^{\infty} e^{-\lambda\,t}\,h(X(t))\,d\,t\Big] = \frac{C(h-\lambda u)}{\lambda\,C(1-\lambda\,v)}\,, \tag{2.46}$$

where $u(x) = \mathbb{E}_x\Big[\int_0^{\tau_0} e^{-\lambda\,s}\,h(X(s))\,d\,s\Big]$ *and* $v(x) = \mathbb{E}_0\Big[\int_0^{\tau_0} e^{-\lambda\,s}\,d\,s\Big]$ *for* $x \in \mathfrak{S}\setminus\{0\}$. *This implies the uniqueness for* $0 < \beta < 2$.

Remark 2.5. Recall that two dimensional wedge is a cone. In Chapter 1, we consider Krein-Rutman theory for the strongly positive compact operator Q on a cone. Fortunately, we can extend computations more explicitly here than there.

2.3 Multidimensional Diffusion with Reflection

In this section we generalize the above results for one dimensional positive half line in Section 2.1 or for two dimensional orthant in Section 2.2 to analyses for the case of $(n-1)-$dimensional orthant $\bar{\mathfrak{S}} := \{x \in \mathbb{R}^{n-1}\,:\, x_i \ge 0\,,\, i = 1,\dots,n-1\}$ with $n \ge 3$. The cases $n = 1, 2$ are covered in Sections 2.1 and 2.2. The boundary $\partial\,\mathfrak{S}$ of $\mathfrak{S}$ has non-smooth part.

In Chapter 1 we consider the $n-$dimensional process $X(\cdot)$. The map $\mathcal{P} : \mathbb{R}^n \mapsto (\mathbb{R}_+)^{n-1}$ in (1.3) transforms the $n-$dimensional process $X(\cdot)$ into the $(n-1)-$dimensional process $\mathcal{P}X(\cdot)$. Our intuition is that this $(n-1)-$dimensional process $\mathcal{P}X(\cdot)$ *behaves like* a multidimensional diffusion with reflection, since $\mathcal{P}X(\cdot)$ cannot move out beyond the positive orthant $(\mathbb{R}_+)^{n-1}$. In Chapter 3 we will consider again the $n-$dimensional process $X(\cdot)$, where $\mathcal{P}X(\cdot)$ behaves *exactly* like the reflected multidimensional diffusion which we describe here. Thus, out goal here is to understand the multidimensional diffusion with reflection on the positive orthant.

Stroock & Varadhan [54] dealt with a class of multidimensional diffusions with reflection in a bounded smooth domain in a general setting where diffusion coëfficients, drift coëfficients and directions of reflection are bounded and continuous functions of the time and space parameter. Tanaka analyzed multi-dimenional diffusions in convex domainwith normal reflection [57]. Lions & Sznitman [43] studied diffusions in non-smooth boundary with non-smooth vector fields which can be approximated by smooth domains and smooth vector fields. Harrison, Reiman & Williams characterized the multidimensional diffusions on the positive orthant in a series of papers [18] [19] [20] [61]. We follow their results and state some extensions of their results.

2.3.1 Multidimensional Skorohod problem

Following Skorohod's approach [52] in Proposition 2.1 of construction of one-dimensional diffusions with reflection, we seek a pair of process $(\Lambda(\cdot)\,, Y(\cdot))$ defined by

$$Y(t) = Y(0) + \xi(t) + (I - \mathfrak{P})\,\Lambda(t) \,\in \mathfrak{S}\,; \quad 0 \le t < \infty\,, \tag{2.47}$$

that satisfy the following conditions below. Here $\xi := \{\xi(t) \,:\, 0 \le t < \infty\}$ is an $(n-1)-$dimensional diffusion on $\mathbb{R}^{n-1}$ starting at zero; the $((n-1) \times (n-1))$ matrix $\mathfrak{P}$ has non-negative elements with zeroes on the diagonal and spectral radius strictly less than one; and I is the identity matrix, as usual. The $i-$th element $\Lambda_i(\cdot)$ of $(n-1)-$dimensional process $\Lambda(\cdot)$ is continuous and nondecreasing with $\Lambda_i(0) = 0$, and Λ_i increases only at those times t where $Y_i(t) = 0$, for $i = 1, \ldots, n-1$. We call the process $Y(\cdot)$ the reflected Brownian motion on $\mathfrak{S}$ generated from $\xi(\cdot)$ or diffusion $\xi(\cdot)$ with reflection. The $((n-1) \times (n-1))$ matrix $I - \mathfrak{P}$ represents the direction of reflection.

The essential question on (2.47 is the following: given the diffusion $\xi(\cdot)$ and the matrix $\mathfrak{P}$ how can we control $\Lambda(\cdot)$ so that $Y(\cdot)$ always stays in the positive orthant $\mathfrak{S}$? The following Proposition 2.3 is due to Harrison & Reiman [18].

Proposition 2.3 (Harrison & Reiman[18])**.** *For the diffusion $\xi(\cdot)$ starting at $\xi(0) \in \mathfrak{S}$, there exists a unique pair $(\Lambda(\cdot)\,, Y(\cdot)\,)$ of continuous processes that satisfy the above conditions. Moreover, we can write*

$$\Lambda(t) = \Psi(\xi(\cdot))\,, \quad Y(t) = (I + \Psi)\,(\xi(t))\,; \quad 0 \le t < \infty\,, \tag{2.48}$$

where Ψ *is a continuous mapping such that*

$$
\begin{aligned}
&\Lambda(t+\theta) - \Lambda(\theta) = \Psi(Y(\theta) + \xi(t+\theta) - \xi(\theta)) \\
&Y(t+\theta) = (I+\Psi)(Y(\theta) + \xi(t+\theta) - \xi(\theta))\,; \quad 0 \le \theta\,, t < \infty\,.
\end{aligned}
\tag{2.49}
$$

Directions of reflection on each face

Let $(e_1, \ldots, e_{n-1})$ be the orthonormal basis of $\mathbb{R}^{n-1}$, where the e_k is $(n-1)-$dimensional vector whose k-th component is equal to one and all other components are equal to zero, for $k = 1, \ldots n-1$. The above reflected Brownian motion reflects on the $(n-2)-$dimensional faces $\mathfrak{F}_1, \ldots, \mathfrak{F}_{n-1}$ of the the non-negative orthant, given as

$$
\mathfrak{F}_i := \left\{ \sum_{k=1}^{n-1} x_k e_k : x_k \ge 0 \text{ for } k = 1, \ldots, n-1,\, x_i = 0 \right\}; \quad 1 \le i \le n-1\,.
\tag{2.50}
$$

Let us denote the $(n-3)-$dimensional faces of intersection by $\mathfrak{F}^o_{ij} := \mathfrak{F}_i \cap \mathfrak{F}_j$ for $1 \le i < j \le n-1$ and their union by $\mathfrak{F}^o := \cup_{1 \le i < j \le n-1} \mathfrak{F}^o_{ij}$. Let $\mathfrak{n}_i$ denote the *inward* unit normal to the face $\mathfrak{F}_i$, and $\mathfrak{q}_i$ be the $i-$th column of $((n-1) \times (n-1))$ matrix $\mathfrak{Q}$, so that $\mathfrak{r}_i := \mathfrak{n}_i + \mathfrak{q}_i$ becomes the direction of reflection of $Y(\cdot)$ in (2.47) on $\mathfrak{F}_i$ for $i = 1, \ldots, n-1$. The $(n-1)-$dimensional vectors $\mathfrak{r}_i$, $\mathfrak{q}_i$ and $\mathfrak{n}_i$ represent the direction of reflection, its tangential component, and its normal component on face $\mathfrak{F}_i$, respectively, for each index $i = 1, \ldots, n-1$. Let us write $\mathfrak{N} := (\mathfrak{n}_1 \ldots \mathfrak{n}_{n-1})$.

2.3.2 Submartingale Problems for Smooth Case

Now let us formulate submartingale problems. Since the positive orthant $\mathfrak{G}$ has non-smooth boundary, it makes our journey longer to take care of the non-smooth part. We first discuss a submartingale problem for a smooth domain and a smooth vector field, following Stroock & Varadhan [54]. Then we approximate the process $Y(\cdot)$, the domain $\mathfrak{G}$ and reflection vector field $\mathfrak{N} + \mathfrak{Q}$. Here is the result for the smooth case.

Theorem 2.4 (Stroock & Varadhan [54])**.** *Given $C^2(\mathbb{R}^n)-$function $\phi : \mathbb{R}^n \mapsto \mathbb{R}$, define a smooth domain G by $G := \{x \in \mathbb{R}^n : \phi(x) > 0\}$. Assume that $\|\nabla\phi\| \ge 1$ on $\partial G = \{x \in \mathbb{R}^n ; \phi(x) = 0\}$. Define bounded continuous functions $a : G \mapsto \mathbb{R}^{n\times n}$, $b : G \mapsto \mathbb{R}^n$ and a locally Lipschitz, bounded vector-valued function $\gamma : \partial G \mapsto \mathbb{R}^n$ with $\langle \gamma\,, \nabla\phi \rangle \ge c$ in ∂G for some positive constant c. Assume that the covariance matrix $A(\cdot) := (\,a_{ij}(\cdot)\,)^n_{i,j=1}$ is strictly* *positive definite. Then, there exists a unique probability measure* $\mathbb{P}$ *on* $(\Omega, \mathcal{F})$*, such that*

1. $\mathbb{P}(Y(t) \in \bar{G}) = 1$

2. *for any* $C_0^2(\mathbb{R}^n)-$*function* $f(\cdot)$ *which satisfies* $\langle \gamma(\cdot), f(\cdot) \rangle \geq 0$ *on* ∂G*, the process defined by*

$$f(Y(t)) - \int_0^t \frac{1}{2} \sum_{i,j=1}^n a_{ij}(Y(s)) \frac{\partial^2 f}{\partial x_i \partial x_j}(Y(s)) \cdot \mathbf{1}_{\{Y(s) \in G\}} \, d\, s\,; \quad 0 \leq t < \infty \tag{2.51}$$

is a $\mathbb{P}-$*submartingale.*

Remark 2.6. In Theorem 2.4 the coëfficients $a(\cdot)$, $b(\cdot)$ and $\gamma(\cdot)$ may depend on the time parameter as well, i.e., $a : [0, \infty) \times G \mapsto \mathbb{R}^{n \times n}$, $b : [0, \infty) \times G \mapsto \mathbb{R}^n$ and $\gamma : [0, \infty) \times \partial G \mapsto \mathbb{R}^n$ with some modifications in (2.51)

Proof of the above Theorem 2.4 consists of two parts, namely, (i) construction of probability measure given by the following recipe in Lemma and (ii) uniqueness of interior process from martingale problem discussed in Chapter 1 and uniqueness of boundary process explained in Lemma 2.8.

Recipe of submartingale problem for the smooth case [54]

- **Càdlàg Process of jump size** δ**.** Given the initial point $y \in G$ of the process $Y(\cdot)$, choose $\delta_0 > 0$ such that $y + \delta\, \gamma(y) \in G$ for $0 \leq \delta \leq \delta_0$. Define a transition probability measure $Q(s, y, t, \Gamma)$ for the Markov process $\{ Q_y^\delta(t) \}$ on the space $D([0, \infty), \bar{G})$ of càdlàg functions $\omega : [0, \infty) \mapsto \bar{G}$, i.e., right continuous functions with left limits, by

$$Q(s, y, t, \Gamma) := \left[\mathbf{1}_\Gamma e^{-(t-s)/\delta} + \delta^{-1} \int_s^t e^{-(u-s)/\delta} \mathbf{1}_\Gamma(y + \delta\, \gamma(y))\, d\, s \right] \mathbf{1}_{\partial G}(y) + \mathbf{1}_{\Gamma \cap G}(y)\,; \tag{2.52}$$

  for $(s,t) \in [0, \infty)^2, y \in \bar{G}, \Gamma \subset \bar{G}$. Then, for $f \in C_b^2(\mathbb{R}^n)$ the process defined by

$$M^Q(t) := f(Y(t)) - \int_0^t \mathbf{1}_{\partial G}(Y(s)) \cdot \delta^{-1} \left[f(y + \delta \gamma(Y(s))) - f(y) \right] d\, s\,; \quad 0 \leq t < \infty\,, \tag{2.53}$$

  is a $Q_x^\delta-$martingale.

- **Patchwork.** For $s \geq 0$ let $\mathbb{P}_{(s,y)}^{(A)}$ be the Markov process on $D([0, \infty), \bar{G})$ associated with the diffusion coëfficients $A(x) = (a_{ij}(x))_{1 \leq i,j \leq n}$ obtain from martingale problem in Chapter 1, and define stopping times $\tau_{-1} = 0$, $\tau_{2\ell} := \inf\{t \ge \tau_{2\ell-1} \,:\, Y(t) \notin G\}$ and $\tau_{2\ell+1} := \inf\{t \ge \tau_{2\ell} \,:\, \|Y(t) - Y(\tau_{2\ell})\| > c\delta \,/\, 2, \}$, recursively. We prepare $\sigma-$fields $\mathcal{M}_{s,t} := \sigma(Y(u)\,, s \le u \le t)$ with completion and paste together a process associated with a sequence of probability measures $\mathbb{P}^{(\cdot)}$ and $\mathbb{P}^{\delta}$ in the following way. Put $\mathbb{P}^{-1} := \mathbb{P}^{(A)}_{(0,x)}$. $\mathbb{P}^{(2\ell)} := \mathbb{P}^{(2\ell-1)}_{(0,x)}$ on $\mathcal{M}(0, \tau_{2\ell-1})$, and given $\mathcal{M}(0, \tau_{2\ell-1})$, require the regular conditional probability distribution of $\mathbb{P}^{(2\ell)}$ is $\mathbb{P}^{(A)}_{\tau_{2\ell-1}, Y(\tau_{2\ell-1})}$. Then, similarly we require $\mathbb{P}^{(2\ell+1)} := \mathbb{P}^{(2\ell)}_{(0,x)}$ on $\mathcal{M}(0, \tau_{2\ell})$, but given $\mathcal{M}(0, \tau_{2\ell})$, require the regular conditional probability distribution of $\mathbb{P}^{(2\ell+1)}$ be $Q^{\delta}_{(\tau_{2\ell}, Y(\tau_{2\ell})}$. Finally, require $\mathbb{P}^{\delta}_{y} = \mathbb{P}^{(\ell)}$ on $\mathcal{M}(0\,,\,\tau_\ell)$.

- **Limit as $\delta \downarrow 0$.** Then, for $f \in C^2_b(\mathbb{R}^n)$ the process defined by

$$
\begin{aligned}
M^{\delta}_f(t) :&= f(Y(t)) - \int_0^t \mathbf{1}_G(Y(s)) \frac{1}{2} \sum_{i,j=1}^{n} \frac{\partial^2 f}{\partial x_i \partial x_j}(Y(s)\, d\, s \\
&\quad - \int_0^t \mathbf{1}_{\partial G}(Y(s)) \cdot \delta^{-1} \left[f(y + \delta \gamma(Y(s))) - f(y) \right] d\, s\,; \quad 0 \le t < \infty
\end{aligned}
\tag{2.54}
$$

is $\mathbb{P}^{\delta}-$submartingale. For any sequence $\{\,\delta_\ell \downarrow 0\}_{\ell=0}^{\infty}$, there exists a probability measure $\mathbb{P}_y$ on $C([0,\infty), \mathbb{R}^n)$ such that $\mathbb{P}^{\delta_\ell}_y \to \mathbb{P}_y$ in $D([0,\infty), \mathbb{R}^n)$ as $\ell \to \infty$. Under $\mathbb{P}_y$, $M_f(\cdot) := \lim_{n\to\infty} M^{\delta_n}_f(\cdot)$ is a submartingale.

$\square$

For $f \in C^2_b(\mathbb{R}^n)$ with $\langle \gamma(\cdot)\,, \nabla f(\cdot) \rangle \ \ge 0$ on ∂G, the process $M_f(\cdot)$ is a submartingale, and hence by the Doob-Meyer decomposition theorem, there is a unique integrable, non-decreasing adapted continuous process $\Lambda_f(\cdot)$ such that $M_f(\cdot) - \Lambda_f(\cdot)$ is a $\mathbb{P}_y-$martingale.

Moreover, take any compact subsets $K (\subset G)$ and $U (\subset G)$ with $K \subset U$. Define $\tau_0 := \inf\{t \ge 0 \,:\, Y(t) \in K\}$, $\tau_{2\ell+1} := \inf\{t \ge \tau_{2\ell} \,:\, X(t) \in \partial U\}$, $\tau_{2\ell} := \inf\{t \ge \tau_{2\ell-1} \,:\, X(t) \in K\}$; $\ell \ge 1$. Then, $M_f(t \wedge \tau_{2\ell+1}) - M_f(t \wedge \tau_{2\ell})$ is a $\mathbb{P}_y-$martingale and hence, so is $\Lambda_f(t \wedge \tau_{2\ell+1}) - \Lambda_f(t \wedge \tau_{2\ell})$, which implies that $\Lambda_f(t \wedge \tau_{2\ell+1}) = \Lambda(t \wedge \tau_{2\ell})$. Thus, the process $\Lambda_f(\cdot)$ only increases at the time t, when $Y(t) \in \partial G$, i.e., $\int_0^t \mathbf{1}_K(Y(s))\, d\,\Lambda_f(s) = 0$. In other words, the process $\Lambda_f(\cdot)$ is characterized by the boundary $\partial G = \{x \in \mathbb{R}^n \,:\, \phi(x) = 0\}$. In fact, we can verify

$$
\frac{d\,\Lambda_f}{d\,\Lambda_\phi}(\cdot) = \frac{\langle\, \gamma\,, \nabla\, f\,\rangle}{\langle\, \gamma\,, \nabla\, \phi\,\rangle}(Y(\cdot))\,; \qquad \text{in } \mathbb{R}_+. \tag{2.55}
$$

This absolutely continuity leads the following Lemma result.

Lemma 2.7 ([54]). *Under the probability measure $\mathbb{P}_y$ pieced together from the above recipe there is a unique, continuous, non-decreasing, adapted process $\Lambda(\cdot)$ such that*

$$\frac{d\,\Lambda}{d\,\Lambda_\phi}(\cdot) = \big[\,\langle\,\gamma\,,\nabla\,\phi\,\rangle\big]^{-1}(Y(\cdot))\,; \qquad \textit{in } \mathbb{R}_+\,, \textit{ and the process}$$

$$f(Y(t)) - \int_0^t \mathbf{1}_G(Y(s))\,\frac{1}{2}\sum_{i,j=1}^n \frac{\partial^2 f}{\partial x_i \partial x_j}(Y(s))\,d\,s - \int_0^t \langle\,\gamma\,,\nabla\,f\,\rangle(Y(s))\,d\,\Lambda(s)\,; \quad 0 \le t < \infty\,,$$

is a $\mathbb{P}_y-$martingale for all $f \in C_b^2(\mathbb{R}^n)$ with $\langle\,\gamma(\cdot)\,,\nabla\,f(\cdot)\,\rangle\ \ge 0$. Moreover, the Laplace transform of $\Lambda(\cdot)$ is finite, i.e., $\mathbb{E}[\,\exp(\,\lambda\Lambda(t)\,)\,] < \infty$ for $\lambda > 0\,,\ t \ge 0$.

Boundary process

Now suppose that $Y(t)$ is on the boundary ∂G and let us consider the process $Y(s)$ afterwards for $s \ge t$. Define a new clock $\tau(\theta) := \sup\{s \ge t\ :\ \Lambda(s) \le \theta\}$ for $\theta \ge 0$, and an $(n+1)-$dimensional process $\eta(\theta) = (\,\tau(\theta)\,, Y(\tau(\theta))\,)$ for $0 \le \theta < \infty$. Let us call $\eta(\cdot)$ the boundary process of $Y(\cdot)$. Note that $\tau(t)$ is not a stopping time with respect to $\mathcal{M}(0,t)$ but an optional time.

In order to look at uniqueness of the boundary process $\eta(\cdot)$, define

$$u(t,y) = \mathbb{E}_y[\,f(\tau(\theta)\,, Y(\tau(\theta))\,]\,; \quad f \in C^2([0,\infty)\times\partial G)\,, \quad t\,,\theta \ge 0 \tag{2.56}$$

Such $u(\,\cdot\,,\,\cdot)$ has bounded continuous special derivative of the first order, which is verified through the theory of partial differential equations [41]. We may verify that

$$f(\eta(t)) - \int_0^t \big(\langle\gamma(Y(\tau(s)))\,, \frac{\partial u}{\partial x}(\tau(s)\,, Y(\tau(s))\,)\,\rangle + \frac{\partial u}{\partial t}(\tau(s)\,, Y(\tau(s))\,)\big)\,d\,s\,; \quad 0 \le t < \infty \tag{2.57}$$

is a $\mathbb{P}_y$-martingale. Thus, we consider a martingale problem

Problem 7 (Martingale Problem for the boundary process). *Given $y \in \partial G$ find a probability measure $\mathbb{P}$ on $D([t,\infty)\times\partial G)$ such that*

1. $\mathbb{P}(\tau(0) = t\,, Y(\tau(0)) = y) = 1\,,$

2. *for any $f \in C^{1,2}([t,\infty)\times\partial G)\,,$*

$$f(\eta(t)) - \int_0^t \big(\langle\gamma(Y(\tau(s)))\,, \frac{\partial u}{\partial x}(\tau(s)\,, Y(\tau(s))\,)\,\rangle + \frac{\partial u}{\partial t}(\tau(s)\,, Y(\tau(s))\,)\big)\,d\,s\,; \quad 0 \le t < \infty$$

is a $\mathbb{P}_y$ martingale to the natural $\sigma-$algebras $\mathcal{M}(0\,,\theta)$. Here let ∞ be an absorbing state, since $\tau(\theta)$ is unbounded.

Lemma 2.8 ([54])**.** *The solution to martingale Problem 7 is unique. Moreover, the solution of submartingale Problem up to exit time from G is unique.*

Then, the submartingale Problem in Theorem 2.4 is well-posed. Therefore, Theorem 2.4 is shown.

2.3.3 Submartingale Problem for Non-smooth Case

In this section we study submartingale problems for the non-smooth case. Our goal is to verify Proposition 2.4, which is a natural generalization of beautiful results obtained by Williams [61].

Let us consider the smooth part $E := \overset{o}{\mathfrak{S}} \cup (\cup_i [\mathfrak{F}_i \backslash \cup_{j \neq i} \mathfrak{F}_{ij}])$ of the positive orthant $\mathfrak{S}$, where $\mathfrak{F}_i$ and $\mathfrak{F}_{ij}$ are defined in (2.50). The difficulty lies in the boundary behavior of diffusion with reflection. We use the results from the previous Section 2.3.2 to approximate non-smoothness. Attainability and recurrence properties play important roles.

Diffusion operator with bounded drift and boundary operator

We consider differential operators $\mathcal{A}\,,\mathcal{D}$ defined by

$$\begin{aligned} &\mathcal{A}\,\phi(x) = \frac{1}{2}\sum_{i,j=1}^{n-1} a_{ij}\cdot \frac{\partial^2\phi(x)}{\partial x_i \partial x_j} + \sum_{i=1}^{n-1} b_i(x)\frac{\partial \phi}{\partial x_i}\,; \quad \phi \in C_b^2(\mathbb{R}) \ \text{ on } \mathfrak{S}\,, \\ &\mathcal{D}\,\phi(x) := \Big\langle \mathfrak{r}_i\,, \nabla\phi \Big\rangle(x) \qquad\qquad\qquad\qquad \text{on } \mathfrak{F}_i\,,\ i = 1, \ldots, n-1\,. \end{aligned} \tag{2.58}$$

We assume that the drift coëfficients $b : \mathbb{R}^{n-1} \mapsto \mathbb{R}^{n-1}$ are bounded, as in Chapter 1, and the $((n-1)\times(n-1))$ variance-covariance matrix $A := (a_{ij})_{1\le i,j\le n-1}$ is fixed and positive-definite. Harrison & Williams [19] [20] [61] studied the case of constant drift coëfficients $b(\cdot) \equiv \bar{b}$. In this section we generalize their results.

The diffusion $\xi(\cdot)$ associated with the differential operator $\mathcal{A}$ is of the form

$$\xi(t) = \xi(0) + \sigma\,B(t) + \int_0^t b(\xi(s))\,d\,s\,; \quad 0 \le t < \infty\,, \tag{2.59}$$

where $((n-1)\times(n-1))$ matrix σ is positive square root of $\sigma\sigma' = A$, and $B(\cdot)$ is an $(n-1)-$dimensional Brownian motion. Since these parameters $(\,\mathfrak{N}, \mathfrak{P}\,, \mathfrak{Q}\,, A\,, b(\cdot)\,)$ affect the property of the diffusion $Y(\cdot)$ with reflection in (2.47) and (2.59), let us call $(\mathfrak{N}, \mathfrak{P}, \mathfrak{Q}, A, b(\cdot))$ the data of the process $Y(\cdot)$.

Remark 2.7 (Rotation and scaling). Let U be a unitary matrix whose columns are orthonormal eigenvectors of the variance-covariance matrix A corresponding to (2.58), and let $\mathfrak{L}$ be the corresponding diagonal matrix of eigenvalues such that $\mathfrak{L} = U'AU$. Note that all the eigenvalues of A are positive. Define $\widetilde{B}(\cdot) := \mathfrak{L}^{-1/2}U\Sigma B(\cdot)$, which is another standard $(n-1)-$dimensional Brownian motion. By rotation and scaling, we can transform the diffusion with reflection into a standard Brownian motion with reflection. Let us define the $((n-1)\times(n-1))$ diagonal matrix with the same diagonal elements as those of A by $\mathfrak{D} = diag(A)$. □

With this preparatory notation, we can state a generalization of submartingale Problem 6 in Section 2.2. Williams [61] shows existence and uniqueness for the submartingale problem below in Proposition 2.4, when the drift coëfficients are constant, i.e., $b(\cdot) \equiv \bar{b}$, and the variance-covariance matrix is the identity matrix, i.e., $A \equiv I$, and moreover, the directions of reflection satisfy the so-called skew-symmetry condition (2.63) below. Once one realizes the idea of Girsanov's transform in Lemma 2.9 below and the skew-symmetry condition in Remark 2.8 below, one can show the statement of Proposition 2.4, if the function $b : \mathbb{R}^{n-1} \mapsto \mathbb{R}^{n-1}$ is any bounded measurable function and the skew-symmetry condition (2.60) holds; See also Remark 2.9.

Proposition 2.4. *Assume that the data $(\mathfrak{N}, \mathfrak{P}, \mathfrak{Q}, A, b(\cdot))$ satisfy the element-wise equations*

$$(2\mathfrak{D} - \mathfrak{P}\mathfrak{D} - \mathfrak{D}\mathfrak{P} - 2A)_{ij} = 0; \quad 1 \le i, j \le n-1, \tag{2.60}$$

where $\mathfrak{D} = diag(A)$, $\mathfrak{P}$ is specified in (2.47)*, $\mathfrak{N}$ and $\mathfrak{Q}$ are directions of reflection explained in* Section 2.3.1*, and moreover assume that $b(\cdot)$ is a bounded measurable function. Then, there is a unique probability measure $\mathbb{P}_y$ on $(\Omega, \mathcal{F})$ such that*

1. $\mathbb{P}_y(Y(0) = y) = 1$.
2. *for each $\varphi \in C_c^2(\bar{\mathfrak{S}})$ that satisfies $\mathcal{D}\varphi \ge 0$ on $\partial\mathfrak{S}$, the process*

$$\varphi(Y(t)) - \int_0^t \mathcal{A}\varphi(Y(s))\,ds; \quad 0 \le t < \infty \tag{2.61}$$

is a $\mathbb{P}_y-$submartingale, and

3. the process never attains the non-smooth part $\mathfrak{S} \setminus E$ *of* $\mathfrak{S}$ *, i.e.,* $\mathbb{P}_y(\tau < \infty) = 0$ *, where*

$$\tau := \inf\{t \geq 0 \,:\, Y(t) \notin E, \} \,. \tag{2.62}$$

Remark 2.8 (Skew symmetric condition). If the covariance matrix is the identity, $A = I$, the equation (2.60) takes the form

$$\mathfrak{N}'\mathfrak{Q} + \mathfrak{Q}'\mathfrak{N} = 0 \,. \tag{2.63}$$

This reduced condition is called *skew-symmetry* condition in [19]. With this observation and Remark 2.7, it is natural to consider the case $A = I$, first and then transform the system by rotation and scaling for the non-identity matrix A . □

To show the above Proposition 2.4, we need preparatory lemmata. First, by an application of idea of Girsanov's theorem we obtain Lemma 2.9.

Lemma 2.9 (Girsanov's change of measure). *Let* $y \in \mathfrak{S}$ *. If a probability measure* $\mathbb{P}_y$ *satisfies the above conditions of Proposition* 2.4*, then*

$$M(t) := \exp\Big(- \int_0^t \sigma^{-1} \, b(Y(s)) \mathbf{1}_{\{Y(s) \in \mathfrak{S}\}} \, d\,B(s) + \frac{1}{2} \int_0^t \| b(Y(s)) \|^2 \, d\,s \Big) \, ; \quad 0 \leq t < \infty \, , \tag{2.64}$$

is a $\mathbb{P}_y-$*martingale and there is a unique probability measure* $\widetilde{\mathbb{P}}_y$ *such that*

$$\frac{d\,\widetilde{\mathbb{P}}_y}{d\,\mathbb{P}_y} = M(t) \, ; \quad 0 \leq t < \infty \, . \tag{2.65}$$

Conversely, if $\widetilde{\mathbb{P}}_y$ *satisfies the above conditions of Proposition* 2.4 *for* $b(\cdot) \equiv 0$ *, then the reciprocal* $[M(\cdot)]^{-1}$ *is a* $\widetilde{\mathbb{P}}_y-$*martingale and there is a unique probability measure* $\mathbb{P}_y$ *with* (2.65)*.*

Remark 2.9. Thus, it suffices to consider the case of zero-drift, i.e., $b(\cdot) \equiv 0$. Moreover, because of Remark 2.8 we can assume the variance-covariance matrix $A \equiv I$. It follows from Theorem 2.3 in Section 2.2 that when $n - 1 = 2$, Proposition 2.4 holds. Proof of Proposition 2.4 is given by mathematical induction.

Let us introduce several families of probability measures, in addition to $\mathbb{P}_y$ and $\widetilde{\mathbb{P}}_y$ in Lemma 2.9, namely, a family $\{\, \mathbb{P}^m \,;\, m \geq 1\}$ of probability measures until some exit times T_m , defined in (2.68), from smooth domains $\mathfrak{S}_m$ in (2.66), its adjoint family $\widehat{\mathbb{P}}^m_y$ of probability measures corresponding to adjoint reflection vector $\widehat{\mathfrak{r}}$ defined in (2.69), and probability measures $\mathbb{P}^\tau$ and $\widehat{\mathbb{P}}^\tau$ for the first exit time τ from the smooth part E defined in Proposition 2.4.

Probability distributions until the exit time from $\mathfrak{S}_m$

Let by $\mathfrak{S}_m$ a sequence of non-empty *bounded* domains with C^3-boundary, such that

$$
\begin{aligned}
&\mathfrak{S}_m \subset \mathfrak{S}_{m+1} \subset \mathfrak{S}\,,\;\; \partial\,\mathfrak{S}_m \cap [\mathfrak{F}_i \setminus \cup_{j\neq i}\mathfrak{F}_{ij}] \neq \emptyset, \\
&\mathfrak{S} = \cup_{m=1}^{\infty}\mathfrak{S}_m\,, \quad \partial\,\mathfrak{S} \cap E = \cup_{m=1}^{\infty}(\partial\,\mathfrak{S} \cap \partial\,\mathfrak{S}_m)\,.
\end{aligned}
\tag{2.66}
$$

Denote by $\mathfrak{n}_m(\cdot)$ the inward unit normal vector field on $\partial\,\mathfrak{S}_m$, and $\mathfrak{r}_m(\cdot) := \sum_{i=1}^{n-1}\mathfrak{r}_i \mathbf{1}_{\mathfrak{F}_i}(\cdot)$ be a $(n\times 1)-$dimensional C^2-vector field on $\partial\mathfrak{S}_m$ such that $\langle \mathfrak{n}_m(\cdot)\,, \mathfrak{r}_m(\cdot)\rangle = 1$, and another $(n\times1)-$dimensional C^2-vector field $\mathfrak{q}_m(\cdot) := \mathfrak{r}_m(\cdot) - \mathfrak{n}_m(\cdot)$ on $\partial\mathfrak{S}_m$ satisfies the skew-symmetry condition, introduced in Remark 2.8,

$$
\langle \mathfrak{n}_m(x)\,, \mathfrak{q}_m(y)\rangle + \langle \mathfrak{q}_m(x)\,, \mathfrak{n}_m(y)\rangle = \mathfrak{n}_m(x)'\,\mathfrak{q}_m(y) + \mathfrak{q}_m(x)'\,\mathfrak{n}_m(y) = 0\,; \quad x\,, y \in \partial\,\mathfrak{S}_m
\tag{2.67}
$$

for $m \geq 1$. Define the stopping times

$$
T_m := \inf\{t \geq 0\,:\, Y(t) \notin \mathfrak{S}_m \cup (\partial\,\mathfrak{S}_m \cap \mathfrak{S})\,\}\,; \quad m \geq 1\,.
\tag{2.68}
$$

Since the boundary $\partial\,\mathfrak{S}_m$ of domain $\mathfrak{S}$, the normal vector field $\mathfrak{n}_m(\cdot)$, the tangential vector field $\mathfrak{q}_m(\cdot)$ and the reflection vector field $\mathfrak{r}_m(\cdot)$, are all smooth, and moreover, the domain $\mathfrak{S}_m$ is bounded, we can apply Theorem 2.4. We obtain a probability measure $\mathbb{P}^m$ on $(\,\Omega\,, \mathcal{F}\,)$ associated with the drift-less Brownian motion on $\bar{\mathfrak{S}}_m$ starting at y with reflection vector field $\mathfrak{r}_m(\cdot)$ on $\partial\,\mathfrak{S}_m$.

Adjoint probability distributions until exit time

Similarly, we can define another family

$$
(\,\widehat{\mathfrak{n}}_m(\cdot) := \mathfrak{n}_m(\cdot)\,, \widehat{\mathfrak{q}}_m(\cdot) := -\mathfrak{q}_m(\cdot)\,, \widehat{\mathfrak{r}}_m(\cdot) := \mathfrak{n}_m(\cdot) - \mathfrak{q}_m(\cdot)\,)
\tag{2.69}
$$

of smooth vector fields on $\partial\,\mathfrak{S}_m$ for $m \geq 1$. They are called the adjoint reflection vector fields, since $\mathfrak{r}_m(\cdot)$ and $\widehat{\mathfrak{r}}_m(\cdot)$ are in adjoint relation. The significance of such adjoint reflection vector fields is studied in Section 2.4. Note that they satisfy the skew-symmetry condition:

(2.70) $$\langle \widehat{\mathfrak{n}}_m(x)\,,\widehat{\mathfrak{q}}_m(y)\rangle + \langle \widehat{\mathfrak{q}}_m(x)\,,\widehat{\mathfrak{n}}_m(y)\rangle = \widehat{\mathfrak{n}}_m(x)'\,\widehat{\mathfrak{q}}_m(y) + \widehat{\mathfrak{q}}_m(x)'\,\widehat{\mathfrak{n}}_m(y) = 0\,; \quad x\,,y \in \partial\,\mathfrak{S}_m\,.$$

Let us define by $\{\widehat{\mathbb{P}}^m_y\}$ the family of probability measure on $(\Omega^\partial\,,\mathcal{F}^\partial)$ associated with the same Brownian motion in $\bar{\mathfrak{S}}_m$ having the adjoint reflection vector $\widehat{\mathfrak{r}}_m(\cdot)$, and by $\widehat{Y}^m(\cdot)$ the realization of Brownian motion with reflection associated with $\{\widehat{\mathbb{P}}^m_y\,,\, y \in \bar{\mathfrak{S}}_m\}$ for $m \geq 1$.

Probability distribution until the first hitting time τ of non-smooth part

Let $E^\partial := E \cup \{\partial\}$ where ∂ is a cemetery point isolated from $\mathfrak{S}$. Similarly, we extend the definitions of the space to $\Omega := \{\omega \in C(E^\partial) : \omega(s) = \partial\,,\, s \geq \tau\}$, $\mathcal{F}^\partial$ be a $\sigma-$field generated by Ω^δ with filtration $\{\mathcal{F}^\delta_t\}_{t\geq 0}$. There is a unique extension $\mathbb{P}^m_y$ on $(\Omega^\partial\,,\mathcal{F}^\partial)$. Then, the family $\{\mathbb{P}^m_y|_{\mathcal{F}^\partial_{T_m}}\,, m \geq 1\}$ of probability measures induces unique probability measures $\mathbb{P}^\tau$ on $(\Omega^\partial\,,\mathcal{F}^\partial)$ such that $\mathbb{P}^\tau_y = \mathbb{P}^m_y$ on $\mathcal{F}^\partial_m$, for $y \in \mathfrak{S}_m$ and for all m sufficiently large. We define $\mathbb{P}^\tau_\partial$ to be the unit mass at the cemetery point ∂. Then, the family $\{\mathbb{P}^\tau_y\,,\, y \in E \cup \partial\}$ has the strong Markov property.

In order to show Proposition 2.4 it suffices to show that the process $Y(\cdot)$ never attains the non-smooth part $\mathfrak{S} \setminus E$ under $\mathbb{P}^\tau_y$, i.e.,

(2.71) $$\mathbb{P}^\tau_y(\tau < \infty) = 0\,; \quad y \in \mathfrak{S}\,.$$

In the same manner, we can introduce $\widehat{\mathbb{P}}^\tau$ associated with the adjoint vector fields.

Size of reflection on the boundary

Assume $b(\cdot) \equiv 0$ and $A \equiv I$, in addition to the assumptions on the skew-symmetry condition (2.63). One of the important points to show (2.71) is to control the size of reflection, since the direction of reflection is given by $\mathfrak{r}_i = \mathfrak{n}_i + \mathfrak{q}_i$ on $\mathcal{F}_i$ for $i = 1, \ldots, n-1$.

The key estimate is the following Lemma 2.10.

Lemma 2.10 ([61]). *Consider the diffusion part $\xi(\cdot)$ defined in (2.59) and the non-decreasing continuous adapted part $\Lambda(\cdot)$ of $Y(\cdot)$. For each $c > 0$, there are $t_0 > 0$ and $\delta \in (0\,,c)$ such* *that for each* $y \in \mathfrak{S}$ *satisfying* $||y|| < \delta$ *, we have*

$$\mathbb{P}_y\big(\max_{0 \le s \le t_0} |\xi(s)| \le c \,, |\Lambda(t_0)| \le c \big) \ge \delta \,. \tag{2.72}$$

In general, t_0 *and* δ *depend on* $c \,, \mathfrak{N} \,, \mathfrak{Q}$ *and the dimension.*

Remark 2.10. When $n - 1 = 1$, it follows from Skorohod equation (2.5) that the continuous non-decreasing process $\Lambda(\cdot)$ can be written as

$$\Lambda(t) = \max[0 \,, \min_{0 \le s \le t} (y + \xi(s))] \le \max_{0 \le s \le t} |\xi(s)| \,. \tag{2.73}$$

Hence, there exists t_0 such that

$$\mathbb{P}_y(\max_{0 \le s \le t_0} |\xi(s)| \le c \,, |\Lambda(t_0)| \le c) = \mathbb{P}_y(\max_{0 \le s \le t_0} |\xi(s)| \le c) > \delta > 0 \,, \tag{2.74}$$

for some positive number δ .

The proof of (2.71) follows by mathematical induction on the dimension $n-1$. Let us *assume* that we can show Lemma 2.10 for the reflected Brownian motion of all dimensions less than or equal to $k (\le n-2)$. Our task is to show Lemma 2.10 for dimension $k+1$. □

Given a point $z \in \bar{\mathfrak{S}} \setminus \{0\}$, there is a sufficiently large m and small $\widetilde{\delta}(z)$ such that an open ball $B_z(\widetilde{\delta}(z))$ of radius $\widetilde{\delta}(z) > 0$ with center at z is contained in $\mathfrak{S}_m$. Then, for $y \in B_z(\widetilde{\delta}(z))$, the process $Y(\cdot)$ behaves like a driftless $(n-1)-$dimensional Brownian motion up to the time $\inf\{ t \ge 0 \, : \, Y(t) \notin B_z(\delta(z)) \} < T_m$ and hence, there is a smaller number $\delta(z) < \widetilde{\delta}(z)$ for a smaller ball $B_z(\delta(z)) \subset B_z(\widetilde{\delta}(z))$ and a positive time $t(z) > 0$ such that

$$\begin{aligned} &\mathbb{P}^{\tau}_y(\, Y(t \wedge \inf\{ t \ge 0 \, : \, Y(t) \notin B_z(\delta(z)) \}) \in E \;\text{ for } t \ge 0 \;) = 1 \,, \\ &\mathbb{P}^{\tau}_y(\, \inf\{ t \ge 0 \, : \, Y(t) \notin B_z(\delta(z)) \} \ge t(z) \;) \ge \delta(z) \, ; \quad y \in B_z(\delta(z)) \,. \end{aligned} \tag{2.75}$$

Let us define a subset $\bar{\mathfrak{S}}_{\varepsilon \,, K} := \{ y \in \bar{\mathfrak{S}} \, : \, \varepsilon \le \|y\| \le K \}$ of $\mathfrak{S}$ for $0 < \varepsilon < K < \infty$ and the first exit time from it:

$$\tau_{\varepsilon \,, K} := \inf\{ t \ge 0 \, : \, Y(t) \notin \bar{\mathfrak{S}}_{\varepsilon \,, K} \} \,. \tag{2.76}$$

It follows from (2.75) that

(2.77)

$$\mathbb{P}^{\tau}_{y}(\,\tau_{1/2\,,2} \ge t(z)\,) \ge \mathbb{P}^{\tau}_{y}(\inf\{t \ge 0 \,:\, Y(t) \notin B_z(\delta(z))\} \ge t(z)\,\,) \ge \delta(z)\,; \quad y \in B_z(\delta(z)) \cap \mathfrak{S}\,.$$

Now take an open cover $\{B_z(\delta(z))\,;\, z \in \bar{\mathfrak{S}}\,$ with $\|z\| = 1\,\}$ of the compact set $\{z \in \bar{\mathfrak{S}}\,$ with $\|z\| = 1\,\}$. By Heine-Borel theorem, there is a finite sub-cover $B_{z_i}\,,\ i = 1, \ldots, \ell$. Taking constants $\delta_0 := \min_{1 \le i \le \ell} \delta(z_i)$ and $t_0 := \min_{1 \le i \le \ell} t(z_i)$, we obtain a uniform estimate of exit probability from $\mathfrak{S}_{1/2\,,2}$:

$$\inf_{\|y\|=1} \mathbb{P}^{\tau}_{y}(\tau_{1/2\,,2} \ge t_0) \ge \delta_0\,. \tag{2.78}$$

Intuitively, this means that with positive $\mathbb{P}^{\tau}_{y}-$probability, the process moves slowly in $\mathfrak{S}_{1/2\,,2}$ in the sense that the reflection near the non-smooth part on the smooth boundary is not so large. Using a scaling property between $\mathbb{P}^{\tau}_{y}$ and $\mathbb{P}^{\tau}_{\lambda\, y}$ for $\lambda > 0$ with this observation, we want to show

$$\mathbb{E}^{\tau}_{y}\Big[\,\exp\Big(-\int_0^{\tau} \|Y(s)\|^{-2}\, d\,s\Big)\,\Big] = 0\,; \quad y \in E\,. \tag{2.79}$$

Then, we obtain

$$\mathbb{P}^{\tau}_{y}(\mathfrak{s}(t) < \tau \ \text{ for } t \ge 0) = 1\,; \quad y \in E \tag{2.80}$$

where $\mathfrak{s}(\cdot)$ is a change of clock that controls the distance $\|Y(\cdot)\|$ of $Y(\cdot)$ from the origin,

$$\Theta(t) := \int_0^t \|Y(s)\|^2\, d\,s\,, \quad \mathfrak{s}(t) := \inf\{s \ge 0 \,:\, \Theta(s) > t\,\}\,. \tag{2.81}$$

In fact, we may show the following Lemma 2.11.

Lemma 2.11 ([61]). *The process always stays inside the smooth part E of $\mathfrak{S}$ up until the stopping time $\tau_{\varepsilon\,,K}$ of (2.76), i.e.,*

$$\mathbb{P}^{\tau}_{y}(Y(t \wedge \tau_{\varepsilon\,,K}) \in E \ \ \textit{for}\ t \ge 0) = 1\,; \quad y \in \mathfrak{S}\,. \tag{2.82}$$

Then, by the scaling property between $\mathbb{P}^{\tau}_{y}$ and $\mathbb{P}^{\tau}_{\lambda\, y}$ for $\lambda > 0$ we obtain another scaling property:

$$\mathbb{E}^{\tau}_{\lambda y}\Big[\exp\Big(-\int_0^{\tau}\|Y(s)\|^{-2}\,d\,s\Big)\Big] = \mathbb{E}^{\tau}_{y}\Big[\exp\Big(-\int_0^{\lambda^2\tau}\|\lambda Y(\lambda^{-2}s)\|^{-2}\,d\,s\Big)\Big]$$
$$= \mathbb{E}^{\tau}_{y}\Big[\exp\Big(-\int_0^{\tau}\|Y(s)\|^{-2}\,d\,s\Big)\Big]$$

and so,

$$\sup_{y\in E}\mathbb{E}^{\tau}_{y}\Big[\exp\Big(-\int_0^{\tau}\|Y(s)\|^{-2}\,d\,s\Big)\Big] = \sup_{\substack{\|y\|=1\\ y\in E}}\mathbb{E}^{\tau}_{y}\Big[\exp\Big(-\int_0^{\tau}\|Y(s)\|^{-2}\,d\,s\Big)\Big]. \tag{2.83}$$

Moreover, by the strong Markov property, we obtain

$$\mathbb{E}^{\tau}_{y}\Big[\exp\Big(-\int_0^{\tau}\|Y(s)\|^{-2}\,d\,s\Big)\Big]$$
$$= \mathbb{E}^{\tau}_{y}\Big[\exp\Big(-\int_0^{\tau_{1/2,2}\wedge t}\|Y(s)\|^{-2}\,d\,s\Big)\cdot\mathbb{E}^{\tau}_{Y(\tau_{1/2,2}\wedge t)}\Big[\exp\Big(-\int_0^{\tau}\|Y(s)\|^{-2}\,d\,s\Big)\Big]\Big].$$

Combining this with (2.83), we obtain

$$\begin{aligned}\sup_{y\in E}\mathbb{E}^{\tau}_{y}\Big[\exp\Big(-\int_0^{\tau}\|Y(s)\|^{-2}\,d\,s\Big)\Big] &= \sup_{\substack{\|y\|=1\\ y\in E}}\mathbb{E}^{\tau}_{y}\Big[\exp\Big(-\int_0^{\tau}\|Y(s)\|^{-2}\,d\,s\Big)\Big]\\ &\le \sup_{\substack{\|y\|=1\\ y\in E}}\mathbb{E}^{\tau}_{y}\Big[\exp\Big(-\int_0^{\tau_{1/2,2}\wedge t}\|Y(s)\|^{-2}\,d\,s\Big)\Big]\cdot\sup_{y\in E}\mathbb{E}^{\tau}_{y}\Big[\exp\Big(-\int_0^{\tau}\|Y(s)\|^{-2}\,d\,s\Big)\Big].\end{aligned} \tag{2.84}$$

It follows now from (2.78) with $\delta(z) < 1/2$ that

$$\begin{aligned}&\sup_{\substack{\|y\|=1\\ y\in E}}\mathbb{E}^{\tau}_{y}\Big[\exp\Big(-\int_0^{\tau_{1/2,2}\wedge t_0}\|Y(s)\|^{-2}\,d\,s\Big)\Big]\\ &\le \sup_{\|y\|=1}\Big[\mathbb{P}^{\tau}_{y}(\tau_{1/2,2}<t_0)+\mathbb{P}^{\tau}_{y}(\tau_{1/2,2}\ge t_0)\,\exp(-t_0/4)\Big]\\ &= \sup_{\|y\|=1}[1-\mathbb{P}^{\tau}_{y}(\tau_{1/2,2}\ge t_0)\,(1-\exp(-t_0/4)]\\ &\le 1-\delta_0\big(1-\exp(-t_0/4)\big) < 1.\end{aligned} \tag{2.85}$$

For both (2.84) and (2.85) to hold, (2.79) must hold as well. Thus, we obtain (2.79) and hence (2.80).

It follows from (2.80) that if the process $Y(\cdot)$ hit the non-smooth part of $\mathfrak{S}$, then the process would hit the origin in the limit as $t \uparrow \tau$, i.e.,

$$\mathbb{P}^{\tau}_{y}(\,\tau < \infty\,) \le \mathbb{P}^{\tau}_{y}(\,\liminf_{t\uparrow\tau}\|Y(t)\| = 0\,)\,; \quad y \in E\,. \tag{2.86}$$

In the following subsections, we rule out the possibility that the process can come to the origin, i.e., we show that the right-hand of (2.86) is zero, using stationary distributions with adjoint relations and Hopf's decomposition theorem. Then, we come back to show Lemma 2.10 for dimension $k+1$ as it is explained in Remark 2.10.

Adjoint stationary distributions

In order to show that the right-hand side of (2.86) is zero, we deploy recurrence properties of Brownian motion with reflection on smooth domains. In Section 2.4 we discuss those properties. Here we explain their consequences.

Let us recall the adjoint Brownian motion $\widehat{Y}(\cdot)$ with the adjoint reflection vector fields (2.69) on the smooth bounded domain $\mathfrak{S}_m$ under the adjoint probability distribution $\widehat{\mathbb{P}}^{m}_{y}$ for $m \ge 1$. We assume that the skew-symmetry condition (2.70) holds for the adjoint vector fields. We extend the definition of $Y(\cdot)$ on Ω^{∂} to get $Z(\cdot)$ defined by

$$Z(t) := Y(\mathfrak{s}(t))\,\mathbf{1}_{\{0\le t<\Theta(\tau-)\}} + \partial\cdot\mathbf{1}_{\{t\ge\Theta(\tau-)\}} \tag{2.87}$$

where $\Theta(\cdot)$ and $\mathfrak{s}(\cdot)$ are defined in (2.80). It follows from Lemma 2.11 that the process $Z(\cdot)$ stays in the smooth part, i.e., $Z(t) = Y(\mathfrak{s}(t)) \in E$ for $0 \le t < \infty$, $y \in E$ under $\mathbb{P}^{\tau}_{y}$. We define $\widehat{Z}(\cdot)$ for $\widehat{\mathbb{P}}^{\tau}_{y}$.

Assume $n-1 \ge 3$, $b(\cdot) \equiv 0$ and $A = I$, because of Remark 2.9. Nagasawa [45] showed that the stationary distributions of the original process Y under $\mathbb{P}^{m}$ and the adjoint process $\widehat{Y}$ under and $\widehat{\mathbb{P}}^{m}$ have duality relative to a Radon measure $\rho(\cdot)$ on $\mathfrak{S}$. The Radon measure $\rho(\cdot)$ has density $\|y\|^{-2}\,d\,y$. In Section 2.4 below it is shown that the stationary distribution of Y under $\mathbb{P}^{m}$ is *uniform* on $\bar{\mathfrak{S}}_m$. See Lemma 2.14. These observations can be applied to the extended processes $Z(\cdot)$ and $\widehat{Z}(\cdot)$, and hence, we obtain the following duality relation:

$$\int_{\mathfrak{S}_m} \mathbb{E}^{m}_{y}\big[\,f(Z(t))\big]\,g(y)\,\|y\|^{-2}\,d\,y = \int_{\mathfrak{S}_m} \widehat{\mathbb{E}}^{m}_{y}\big[\,g(Z(t))\big]\,f(y)\,\|y\|^{-2}\,d\,y; \quad t \ge 0\,,\; y \in \mathfrak{S}_m\,, \tag{2.88}$$

for all continuous functions $f(\cdot)$ and $g(\cdot)$ having compact support in $\mathbb{R}^{n-1}$. By taking limits on both sides as $m \to \infty$, we obtain

$$\int_{\mathfrak{S}} \mathbb{E}_y \big[f(Z(t)) \big] g(z) \, \|y\|^{-2} \, d\, y = \int_{\mathfrak{S}} \widehat{\mathbb{E}}_y \big[g(Z(t)) \big] g(y) \, \|y\|^{-2} \, d\, y; \quad t \ge 0 \,, \; y \in E \,, \tag{2.89}$$

for the functions $f(\cdot)$ and $g(\cdot)$ in same classes of functions in (2.88).

An application of Hopf's decomposition theorem

This (2.89) shows that the Radon measure $\rho(\cdot)$ is an invariant measure for a Markov chain $w(\omega) := \{ w(i) := w(i, \omega) \in \mathfrak{S} \,, i = 0, 1, 2, \ldots \}$ on $(\Omega, \sigma(\mathfrak{S}^{\mathbb{N}}))$ with one-step transition probabilities $Q(\, y \,, \, d\, z \,) := \mathbb{P}^{\tau}[w(i+1) \in \, d\, z \,|\, w(i) = y]$ for $(y, z) \in \mathfrak{S}^2$, $i = 0, 1, \ldots$. Let us define the probability distribution on the path space $\mathfrak{S}^{\mathbb{N}}$ of the Markov chain $w := \{w(i)\}$ with the initial distribution $\rho(\cdot)$ by $\mathbb{P}_{\rho}$, a nonnegative $\rho-$integrable functional $\varphi : \mathfrak{S}^{\mathbb{N}} \mapsto \mathbb{R}_+$ on the path space $\mathfrak{S}^{\mathbb{N}}$, i.e., $\varphi(w) = \varphi(w(0), w(1), \ldots)$, with $\mathbb{E}_{\rho}(\varphi(w)) < \infty$, and the shift operator $\mathfrak{T}$ on the class of such functionals : $\mathfrak{T}\varphi(w) = \varphi(w(1), w(2), \ldots)$.

Here we use the following consequence of Hopf's decomposition theorem.

Theorem 2.5 (Theorem 2.3 of Revuz [50])**.** *For the shift operator $\mathfrak{T}$, there exists a* conservative *subset $\mathcal{C}$ of $\mathfrak{S}^{\mathbb{N}}$, unique up to equivalence, such that for every non-negative $\rho-$integrable functional φ, we have*

$$\sum_{i=0}^{\infty} \mathfrak{T}^i \varphi(w) = 0 \;\; \textit{or} \; \infty \;\; \textit{if} \; w \in \mathcal{C} \,, \quad \sum_{i=0}^{\infty} \mathfrak{T}^i \varphi(w) < \infty \;\; \textit{if} \; w \in \mathcal{C}^c \,. \tag{2.90}$$

Take $\varphi_1(w) := \mathbf{1}_{B_0(1)}(w(0))$ in $\mathcal{C}$ and $\varphi_2(w) := \mathbf{1}_{B_0(1) \setminus B_0(1\,/\,j)}(w(0))$ for $j \ge 2$ in $\mathcal{C}^c$, where $B_x(r)$ is the ball of radius r with center at x. Those functionals are $\rho-$integrable if $n - 1 \ge 3$, i.e., $\mathbb{E}_{\rho}[\varphi_{\ell}(w)] < \infty$ for $\ell = 1, 2$. Then, we obtain

$$\begin{aligned} &\sum_{\ell=0}^{\infty} \mathbf{1}_{B_0(1)}(w(i)) = 0 \;\text{ or } \infty \;\text{ on } \mathcal{C} \\ &\sum_{\ell=0}^{\infty} \mathbf{1}_{B_0(1) \setminus B_0(1\,/\,j)}(w(i)) < \infty \;\text{ on } \mathcal{C}^c \,; \quad j \ge 2 \,. \end{aligned} \tag{2.91}$$

It follows that $\mathbb{P}_{\rho}(\limsup_{i \to \infty} \|w(i)\| = 0) = 0$ and hence

$$\mathbb{P}^{\tau}_y (\limsup_{i \to \infty} \|Z(i)\| = 0) = 0 \;\, \rho - a.e. \, y \, \in \mathfrak{S} \,. \tag{2.92}$$

Now let us recall the definition of $\tau_{\varepsilon\,,K}$ for $0 < \varepsilon < K < \infty$ in (2.76). By the scaling property and (2.78) we have

$$\delta_0 \le \inf_{\substack{z\in E\\ |z|=1\,/\,j}} \mathbb{P}^{\tau}_{z}(\tau_{1\,/(2j)\,,2\,/\,j} \ge j^{-2}\,t_0\,)\,. \tag{2.93}$$

Defining recursively the stopping times $\sigma_1 := \inf\{t \ge 0 \,:\, Y(t) \notin B_0(1/j)\cap E\,\}$, $\tau_1 := \inf\{t \ge \sigma_1 \,:\, Y(t) \in E \setminus B_0(1\,/\,(2j)\,)\,\}$, $\sigma_i := \inf\{t \ge \tau_{i-1} \,:\, Y(t) \notin B_0(1/j)\cap E\,\}$, $\tau_i := \inf\{t \ge \sigma_{i-1} \,:\, Y(t) \in E \setminus B_0(1\,/\,(2j)\,)\,\}$ for $i \ge 2\,,\ j \ge 2$, we obtain from the Strong Markov property

$$\sum_{i=1}^{\infty} \mathbf{1}_{\{\sigma_i<\infty\}} \mathbb{P}^{\tau}_{Y(\sigma_i)}(\,\tau_{1\,/\,(2j)\,,\,2\,/\,j} \ge t\,) \le \sum_{i=1}^{\infty} \mathbb{P}^{\tau}_{y}(\,\sigma_i < \infty\,, \tau_i - \sigma_i \ge t\,|\,\mathcal{F}_{\sigma_i}\,). \tag{2.94}$$

Then combining with (2.93) and an extension of the Borel-Cantelli lemma, we get

$$\begin{aligned}
&\{\,\liminf_{t\to\tau}\|Y(t)\| = 0\,\} \cap \{\limsup_{t\to\tau}\|Y(t)\| \ge 1\,/\,j\,\}\\
\subset&\Big\{\sum_{i=1}^{\infty} \mathbb{P}^{\tau}_{y}(\sigma_i < \infty\,, \tau_i - \sigma_i \ge j^{-2} t_0\,|\,\mathcal{F}_{\sigma_i}) = \infty\Big\}\\
&= \{\sigma_i < \infty\,, \tau_i - \sigma_i \ge j^{-2} t_0 \ \text{ i.o. in } i\,\} \subset \{\,\tau = \infty\,\}\,; \quad \mathbb{P}^{\tau}_{y}\text{-}a.s.
\end{aligned}$$

With this and (2.92) together yield

$$\{\,\liminf_{t\to\infty}\|Y(t)\| = 0\,\} = \cup_{j=1}^{\infty}[\,\{\liminf_{t\to\infty}\|Y(t)\| = 0\,\} \cap \{\limsup_{t\to\infty}\|Y(t)\| \ge 1\,/\,j\,\}\,] \subset \{\,\tau = \infty\,\}\,;$$

$\mathbb{P}^{\tau}_{y}$-$a.s.$ and $\rho - a.e.\, y \in E$. Combining this with (2.86), we obtain $\{\,\tau < \infty\,\} \subset \{\,\tau = \infty\,\}$ mod $\mathbb{P}^{\tau}_{y}$. This is not a contradiction only if $\mathbb{P}^{\tau}_{y}(\tau < \infty) = 0$ for $y \in \mathfrak{S}$. Let us summarize the above argument.

Lemma 2.12 ([61])**.** *Assume $b(\cdot) \equiv 0$ and $A \equiv I$, and moreover assume that Lemma* 2.10 *is true for all dimensions less than or equal to $k(\le n-2)$. Then,* (2.71) *holds for the* $(k+1)-$*dimensional Brownian motion with reflection never attain the non-smooth part of $\mathfrak{S}$, i.e., $\mathbb{P}^{\tau}_{y}(\,\tau < \infty\,) = 0$ for $y \in \mathfrak{S} \subset \mathbb{R}^{k+1}$.*

2.4 Ergodicity

2.4.1 Smooth Bounded Domain

Let $\mathfrak{S}$ be a nonempty bounded domain in $\mathbb{R}^{n-1}$ of class $C^{2+\alpha}$. Let $\mathfrak{n}(\cdot)$ denote the inward unit normal vector field on the boundary $\partial\mathfrak{S}$ of $\mathfrak{S}$. Let us define the reflection vector field $\mathfrak{r}(\cdot)$ of $C^{1+\alpha}$ such that $\langle \mathfrak{r}(\cdot)\,, \mathfrak{n}(\cdot)\rangle = 1$. The vector filed $\mathfrak{q}(\cdot) := \mathfrak{r}(\cdot) - \mathfrak{n}(\cdot)$ is the tangential component of $\mathfrak{r}(\cdot)$ on $\partial\mathfrak{S}$. The drift coëfficient $b(\cdot)$ is assumed to be a constant $\bar{b}$ in this Section 2.4.1. We see a generalization in Section 2.4.2.

Differential operators

We define the differential operator $\mathcal{A}_1$ which corresponds to the $(n-1)-$dimensional Brownian motion with drift rate $\bar{b}$ in the bounded smooth domain $\mathfrak{S}$, and the differential operator $\mathcal{D}^1$ on the boundary $\partial\mathfrak{S}$ by

$$\begin{aligned} &\mathcal{A}_1\varphi(\cdot) = \frac{1}{2}\Delta\varphi(\cdot) + \langle \bar{b}\,, \nabla\varphi(\cdot)\rangle \quad \text{in } \mathfrak{S}\,, \\ &\mathcal{D}^1\varphi(\cdot) = \langle \mathfrak{r}(\cdot)\,, \nabla\varphi(\cdot)\rangle = \langle \mathfrak{n}(\cdot)\,, \nabla\varphi(\cdot)\rangle + \langle \mathfrak{q}(\cdot), \nabla\varphi - \langle \mathfrak{n}(\cdot)\,, \nabla\varphi\rangle\mathfrak{n}(\cdot)\rangle \quad \text{on } \partial\,\mathfrak{S}\,, \end{aligned} \tag{2.95}$$

where Δ is the $(n-1)-$dimensional Laplacian operator for $\varphi(\cdot) \in C^2(\bar{\mathfrak{S}})$ Their adjoint operators $\widehat{\mathcal{A}}_1$ and $\widehat{\mathcal{D}}^1$ are defined by

$$\begin{aligned} &\widehat{\mathcal{A}}_1\varphi(\cdot) := \frac{1}{2}\Delta\varphi(\cdot) - \langle \bar{b}\,, \nabla\varphi(\cdot)\rangle \quad \text{in } \mathfrak{S}\,, \\ &\widehat{\mathcal{D}}^1\varphi(\cdot) := \langle \widehat{\mathfrak{r}}(\cdot)\,, \nabla\varphi(\cdot)\rangle := \langle \mathfrak{n}(\cdot)\,, \nabla\varphi(\cdot)\rangle - \langle \mathfrak{q}(\cdot), \nabla\varphi - \langle \mathfrak{n}(\cdot)\,, \nabla\varphi\rangle\mathfrak{n}(\cdot)\rangle \quad \text{on } \partial\,\mathfrak{S}\,. \end{aligned} \tag{2.96}$$

Remark 2.11. The above set-up makes it possible to use the following result on the Dirichlet problem with oblique reflection in smooth bounded domain. For the non-smooth domain we take another approach to get the basic adjoint relation in Section 2.4.2. □

Problem 8 (Dirichlet Problem with oblique reflection [16]). *For every $h(\cdot) \in C^{\alpha}(\bar{\mathfrak{S}})$, $\lambda > 0$, there exists $f(\cdot) \in C^{2+\alpha}(\bar{\mathfrak{S}})$ such that*

$$\mathcal{A}_1\, f(\cdot) = \lambda f(\cdot) - h(\cdot) \quad \textit{in } \mathfrak{S}\,, \quad \mathcal{D}^1\, f(\cdot) = 0 \quad \textit{on } \mathfrak{S}\,. \tag{2.97}$$

This is true for the adjoint operators $\widehat{\mathcal{A}}_1$ and $\widehat{\mathcal{D}}^1$, i.e., there exists the solution $\widehat{f} \in C^{2+\alpha}(\mathfrak{S})$.

Weiss [60] was the first to show that the Brownian motion with reflection on the bounded smooth domain has a unique stationary distribution. In this section we show that under the skew-symmetry condition

$$\langle \mathfrak{n}(x)\,,\,\mathfrak{q}(y)\rangle + \langle \mathfrak{q}(x)\,,\,\mathfrak{n}(y)\rangle = 0\,; \quad x, y \in \partial\mathfrak{G}\,, \tag{2.98}$$

the stationary distribution has a density of exponential form.

Basic adjoint relation

In Proposition 2.4 of Section 2.3.2 we saw the $(n-1)-$dimensional Brownian motion with smooth reflection vector field on the smooth domain is well defined as the solution of the submartingale problem. Let us denote by $\mathbb{P}_y$ the solution of the submartingale corresponding to the differential operators $\mathcal{A}_1$ and $\mathcal{D}_1$ in (2.95) with the initial point $y \in \mathfrak{G}$. By the submartingale characterization,

$$f(Y(t)) - \int_0^t \mathcal{A}_1 f(Y(s))\,d\,s\,; \quad 0 \le t < \infty \tag{2.99}$$

is a $\mathbb{P}_y-$martingale for every $f \in C^{2+\varepsilon}(\mathbb{R}^n)$ with $\mathcal{D}^1 f = 0$ on $\partial\mathfrak{G}$. Note that because of the boundary condition $\mathcal{D}^1 f = 0$ on $\partial\mathfrak{G}$, it is *not only* a submartingale but a martingale.

Take the solution $f(\cdot)$ in (2.97) of Dirichlet problem 8 in Remark 2.11 for $\lambda > 0$ and $h \in C^{\alpha}(\mathbb{R}^{n-1})$. By Itô's formula,

$$e^{-\lambda\,t} f(Y(t)) - \int_0^t e^{-\lambda\,s}(-\lambda f + \mathcal{A}_1 f)(Y(s))\,d\,s = e^{-\lambda\,t} f(Y(t)) - \int_0^t e^{-\lambda\,s} h(Y(s))\,d\,s \tag{2.100}$$

is another $\mathbb{P}_y-$martingale starting at $f(y)$. Then, taking the expectations and letting $t \to \infty$ we obtain the resolvent operator

$$f(y) = \mathbb{E}_y\Big[\int_0^\infty e^{-\lambda\,s} h(Y(s))\,d\,s\Big] \equiv R_\lambda h(y)\,. \tag{2.101}$$

The stationary distribution has density function $\mathfrak{p}(\cdot)$ if and only if

$$\int_{\bar{\mathfrak{G}}} \mathbb{E}_y[h(Y(t))]\mathfrak{p}(y)\,d\,y = \int_{\bar{\mathfrak{G}}} h(y)\mathfrak{p}(y)\,d\,y\,; \quad h \in C^{\alpha}(\bar{\mathfrak{G}})\,,\, t \ge 0\,. \tag{2.102}$$

This is equivalent to

$$\lambda \int_{\bar{\mathfrak{S}}} f(y)\, \mathfrak{p}(y)\, d\, y = \lambda \int_{\bar{\mathfrak{S}}} R_\lambda h(y) \mathfrak{p}(y)\, d\, y = \int_{\bar{\mathfrak{S}}} h(y)\, \mathfrak{p}(y)\, d\, y\,, \tag{2.103}$$

because

$$\begin{aligned} &\int_0^\infty d\, s e^{-\lambda\, s} \int_{\bar{\mathfrak{S}}} \mathbb{E}_y (h(Y(s))) \mathfrak{p}(y)\, d\, y = \int_{\bar{\mathfrak{S}}} d\, y\, \mathfrak{p}(y) R_\lambda(y) \\ &= \int_0^\infty d\, s\, e^{-\lambda\, s} \int_{\bar{\mathfrak{S}}} h(y)\, \mathfrak{p}(y)\, d\, y = \lambda^{-1} \int_{\bar{\mathfrak{S}}} h(y)\, \mathfrak{p}(y)\, d\, y\,. \end{aligned} \tag{2.104}$$

Thus, it follows from (2.103) that for $f \in C^{2+\alpha}(\bar{\mathfrak{S}})$ with $\mathcal{D}^1 f = 0$ on $\partial\, \mathfrak{S}$, the density function $\mathfrak{p}(\cdot)$ of the stationary distribution satisfies

$$\int_{\bar{\mathfrak{S}}} [\mathcal{A}_1 f(y)]\, \mathfrak{p}(y)\, d\, y = \int_{\bar{\mathfrak{S}}} (\lambda f - h)(y)\, \mathfrak{p}(y)\, d\, y = 0\,. \tag{2.105}$$

Conversely, by the uniqueness of the inverse Laplace transform, if a nonnegative function $\mathfrak{p}(\cdot)$ satisfies (2.105), then $\mathfrak{p}(\cdot)$ is the probability density function of the stationary distribution. Thus, (2.105) characterizes the density function $\mathfrak{p}(\cdot)$.

By Green's theorem and the divergence theorem, we obtain

(2.106)

$$\begin{aligned} &\int_{\bar{\mathfrak{S}}} [\mathcal{A}_1\, f(y)]\, \mathfrak{p}(y)\, d\, y = \int_{\mathfrak{S}} f(y)\, \widehat{\mathcal{A}}_1\, \mathfrak{p}(y)\, d\, y + \frac{1}{2} \int_{\partial \mathfrak{S}} \Big(f\, \frac{\partial \mathfrak{p}}{\partial \mathfrak{n}} - \mathfrak{p}\, \frac{\partial f}{\partial \mathfrak{n}} - 2 \langle \bar{b}\,,\, \mathfrak{n} \rangle \mathfrak{p}\, f \Big)(y)\, v(\, d\, y\,) \\ &= \int_{\mathfrak{S}} [\widehat{\mathcal{A}}_1 \mathfrak{p}(y)]\, f(y)\, d\, y \\ &\qquad + \frac{1}{2} \int_{\partial \mathfrak{S}} \Big[\Big(\frac{\partial \mathfrak{p}}{\partial \mathfrak{n}} - \mathfrak{q} \cdot \nabla_T \mathfrak{p} - (\nabla_T \cdot \mathfrak{q} + 2 \langle \bar{b}\,,\, \mathfrak{n} \rangle) \Big)(y)\, f(y) - \mathfrak{p}(y)\, \mathcal{D}^1\, f(y) \Big]\, v(\, d\, y\,) \\ &= \int_{\mathfrak{S}} [\widehat{\mathcal{A}}_1 \mathfrak{p}(y)]\, f(y)\, d\, y + \frac{1}{2} \int_{\partial \mathfrak{S}} \Big(\widehat{\mathcal{D}}^1\, \mathfrak{p}\, - (\nabla_T \cdot \mathfrak{q} + 2 \langle \bar{b}\,,\, \mathfrak{n} \rangle) \Big)(y)\, f(y)\, v(\, d\, y\,)\,; \quad f \in C^{2+\alpha}\,, \end{aligned}$$

from (2.105) where for simplicity we use the notations

$$\frac{\partial \varphi}{\partial \mathfrak{n}}(\cdot) := \langle \mathfrak{n}(\cdot)\,,\, \nabla \varphi(\cdot) \rangle\,, \quad \nabla_T \varphi(\cdot) := \nabla \varphi(\cdot) - \mathfrak{n}(\cdot)\, \langle \mathfrak{n}(\cdot)\,,\, \nabla \varphi(\cdot) \rangle\,; \quad \varphi \in C^{1+\alpha}(\mathfrak{S})\,, \tag{2.107}$$

$\nabla_T \cdot \mathfrak{q}$ is the divergence of $\mathfrak{q}(\cdot)$ on the boundary $\partial\, \mathfrak{S}$, and $v(\, d\, y\,)$ is the $(n-2)-$dimensional surface measure. Thus, (2.105) is equivalent to

$$\widehat{\mathcal{A}}_1\, \mathfrak{p}(\cdot) = 0 \;\text{ in } \mathfrak{S}\,, \quad \widehat{\mathcal{D}}_1\, \mathfrak{p}(\cdot) = (\nabla_T \cdot \mathfrak{q}(\cdot) + 2 \langle \bar{b}\,,\, \mathfrak{n}(\cdot) \rangle)\, \mathfrak{p}(\cdot) \;\text{ on } \partial\, \mathfrak{S}\,. \tag{2.108}$$

Moreover, this is equivalent to the *basic adjoint relation* in [20]:

$$\int_{\mathfrak{S}} [\mathcal{A}_1 f(y)]\mathfrak{p}(y)\, d\, y + \frac{1}{2}\int_{\partial\mathfrak{S}} \mathfrak{p}(y)\, \mathcal{D}^1 f(y)\, v(\, d\, y\,) = 0\,; \quad f \in C_0^2(\bar{\mathfrak{S}})\,. \tag{2.109}$$

Now we use the following Lemma 2.13 to obtain a solvable

$$\widehat{\mathcal{A}}_1\, \mathfrak{p}(\cdot) = 0 \ \text{ in } \mathfrak{S}\,, \quad \widehat{\mathcal{D}}^1\, \mathfrak{p}(\cdot) = 2\langle \bar{b}\,,\, \mathfrak{n}(\cdot)\rangle\, \mathfrak{p}(\cdot) \ \text{ on } \partial\,\mathfrak{S}\,. \tag{2.110}$$

Lemma 2.13 ([20])**.** *The skew-symmetry condition* (2.98) *implies* $\nabla_T \cdot \mathfrak{q}(\cdot) = 0$ *on* $\partial\,\mathfrak{S}$ *where* $\mathfrak{q}(\cdot)$ *is the vector field tangential to the reflection vector field* $\mathfrak{r}(\cdot)$ *.*

In fact, for each $y \in \partial\,\mathfrak{S}$, consider an orthonormal system $o_1, \ldots, o_{n-2}$ to the boundary $\partial\,\mathfrak{S}$ at y and the covariant derivative $\mathcal{D}_{o_i}$ in the direction o_i for $i = 1, \ldots, n-2$ [21]. By the definition of divergence $\nabla_T \cdot \mathfrak{q}$ of the vector field $\mathfrak{q}(\cdot)$ on the boundary at y is

$$\nabla_T \cdot \mathfrak{q}(y) = \sum_{i=1}^{n-2} \langle \mathcal{D}_{o_i}\, \mathfrak{q}(y)\,,\, o_i\rangle = \sum_{i=1}^{n-2} \langle \mathcal{D}_{o_i}\, \mathfrak{q}(y)\,,\, \mathfrak{n}(y_i^*)\,\rangle\,, \tag{2.111}$$

where $y_i^* \in \partial\,\mathfrak{S}$ is chosen so that $o_i = \mathfrak{n}(y_i^*)$ for $i = 1, \ldots, n-2$. The skew-symmetry condition (2.98) implies

$$\begin{aligned} &\langle \mathcal{D}_{o_i}\, \mathfrak{q}(y)\,,\, \mathfrak{n}(y_i^*)\,\rangle = -\langle \mathfrak{q}(y_i^*)\,,\, \mathcal{D}_{o_i}\mathfrak{n}(y)\,\rangle\,, \\ &\langle \mathfrak{q}(y_i^*)\,,\, o_j\,\rangle = \langle \mathfrak{q}(y_i^*\,,\, \mathfrak{n}(y_j^*)\,\rangle = -\langle \mathfrak{n}(y_i^*)\,,\, \mathfrak{q}(y_j^*)\,\rangle = -\langle o_i\,,\, \mathfrak{q}(y_j^*)\,\rangle\,. \end{aligned} \tag{2.112}$$

Since $\mathcal{D}_{o_i}\mathfrak{n}(y)$ lies in the tangent space to $\partial\,\mathfrak{S}$ at y, it can be written as $\mathcal{D}_{o_i}\mathfrak{n}(y) = \sum_{j=1}^{n-2} c_{ij}(y) o_j$ for some coëfficients $c_{ij}(y)$ which are symmetric with $c_{ii}(y) = 0$ in the second fundamental form; See [21]. Then, we obtain

$$\begin{aligned} \nabla_T \cdot \mathfrak{q}(y) &= \sum_{i=1}^{n-2} \langle \mathcal{D}_{o_i}\mathfrak{q}(y)\,,\, \mathfrak{n}(y_i^*)\,\rangle = -\sum_{i=1}^{n-2} \langle \mathcal{D}_{o_i}\mathfrak{n}(y)\,, \mathfrak{q}(y_i^*)\,\rangle \\ &= -\sum_{i=1}^{n-2}\sum_{j=1}^{n-2} c_{ij}(y) \langle o_j\,,\, \mathfrak{q}(y_i^*)\,\rangle = \sum_{i=1}^{n-2}\sum_{j=1}^{n-2} c_{ij}(y)\langle o_i\,,\, \mathfrak{q}(y_j^*)\,\rangle \\ &= \sum_{i=1}^{n-2}\sum_{j=1}^{n-2} c_{ji}(y)\langle o_j\,, \mathfrak{q}(y_i^*)\,\rangle = 0\,, \end{aligned} \tag{2.113}$$

since the last summation consists of skew-symmetry summands. Thus, Lemma 2.13 is shown and hence we obtain (2.110).

An *Ansatz* for the solution $\mathfrak{p}(\cdot)$ of (2.110) is of the form $\mathfrak{p}(y) = c\exp(\langle\gamma\,, y\rangle)$ for $y \in \mathfrak{S}$ and some vector $\bar{\gamma} \in \mathbb{R}^{n-1}$ with some constant c. Substituting this into (2.110), with (2.96) we obtain

$$\frac{1}{2}\|\bar{\gamma}\|^2 - \langle\bar{b}\,,\bar{\gamma}\rangle = 0\,, \quad \langle(\mathfrak{n}(\cdot) - \mathfrak{q}(\cdot)\,,\,\bar{\gamma}\rangle = \nabla_T \cdot \mathfrak{q}(\cdot) + 2\langle\bar{b}\,,\,\mathfrak{n}(\cdot)\rangle \ \text{ on } \partial\,\mathfrak{S}\,. \tag{2.114}$$

If the drift vector is zero, i.e., $\bar{b} = 0$, then so is $\bar{\gamma}$. Thus, we obtain the following Lemma 2.14. This fact is explained with relation to adjoint stationary distributions $\mathbb{P}^{\tau}$ and $\widehat{\mathbb{P}}^{\tau}_{y}$ in Section 2.3.3 already.

Lemma 2.14 ([20])**.** *The no-drift* ($\bar{b} = 0$) *Brownian motion with the above smooth reflection* $\mathfrak{r}(\cdot)$ *in the smooth bounded domain* $\mathfrak{S}$ *has uniform stationary distribution.*

If the drift vector is non-zero, we choose some points $y^{(1)}, \dots, y^{(n-1)}$ on $\partial\,\mathfrak{S}$ such that $\mathfrak{n}(y^{(1)})\,, \dots, \mathfrak{n}(y^{(n-1)})$ are linearly independent. Define $((n-1)\times(n-1))$ matrix whose ith column is $\mathfrak{n}(y^{(i)})$ (respectively $\mathfrak{q}(y^{(i)})$) by $\bar{\mathfrak{N}}$ (respectively $\bar{\mathfrak{Q}}$). Then, (2.114) holds only if

$$(I - \bar{\mathfrak{N}}^{-1}\,\bar{\mathfrak{Q}})\,\bar{\gamma} = 2\bar{b}\,. \tag{2.115}$$

If $I - \bar{\mathfrak{N}}^{-1}\bar{\mathfrak{Q}}$ is invertible, then

$$\bar{\gamma} = 2(I - \bar{\mathfrak{N}}^{-1}\,\bar{\mathfrak{Q}})^{-1}\bar{b}\,, \tag{2.116}$$

and hence the first equation in (2.114) implies $\bar{\gamma}\,\bar{\mathfrak{N}}^{-1}\,\bar{\mathfrak{Q}}\,\bar{\gamma} = 0$. This holds for all $\bar{\gamma} \in \mathbb{R}^{n-1}$ if and only if $\bar{\mathfrak{N}}^{-1}\,\bar{\mathfrak{Q}}$ is skew-symmetry. Conversely, if $\bar{\mathfrak{N}}^{-1}\,\bar{\mathfrak{Q}}$ is skew-symmetric, then $I - \bar{\mathfrak{N}}^{-1}\,\bar{\mathfrak{Q}}$ is invertible. This is equivalent to $\bar{\mathfrak{N}}'\,\bar{\mathfrak{Q}}$ being skew-symmetric. It can be shown that if there are two $((n-1)\times(n-1))$ matrix $\bar{\mathfrak{Q}}$ and $\bar{\mathfrak{N}}$ satisfy the skew-symmetry condition $\bar{\mathfrak{Q}}'\bar{\mathfrak{N}} + \bar{\mathfrak{N}}'\bar{\mathfrak{Q}} = 0$, then there exist a unique vector fields $\mathfrak{n}(\cdot)$ and $\mathfrak{q}(\cdot)$ satisfies the skew-symmetry condition (2.98) on the boundary $\partial\,\mathfrak{S}$. Moreover, if (2.98) holds, $\bar{\mathfrak{N}}^{-1}\,\bar{\mathfrak{Q}}$ and hence $\bar{\gamma}$ in (2.116) is independent of the particular choice of $\bar{\mathfrak{N}}$ and $\bar{\mathfrak{Q}}$. Thus, with these observations, we obtain the following Proposition 2.5.

Proposition 2.5 (Harrison & Williams [20])**.** *Given a fixed bounded* $C^{2+\alpha}$ *domain* $\mathfrak{S}$ *and* $C^{1+\alpha}$*−reflection vector field* $\mathfrak{r}(\cdot)$ *on* $\partial\,\mathfrak{S}$*, the following conditions for the reflected Brownian motion with drift vector* $\bar{b} \in \mathbb{R}^{n-1}$ *are equivalent.*

- *The stationary distribution of the reflected Brownian motion has the exponential form density*

$$\mathfrak{p}(y) = c \exp(\langle \bar{\gamma}, y \rangle); \quad y \in \mathfrak{S}, \tag{2.117}$$

where $\bar{\gamma} \in \mathbb{R}^{n-1}$ *and* $c > 0$ *are determined by* $\bar{b}$ *through* (2.116) *and*

$$c := \Big(\int_{\mathfrak{S}} \exp\big(\langle \bar{\gamma}, y \rangle\big)\, d\, y \Big)^{-1}. \tag{2.118}$$

- *The reflection vector field satisfies the skew-symmetry condition* (2.98)*.*

In Section 2.4.2 we extend these considerations and see in Proposition 2.5 that a stationary distribution for the process with piece-wise constant drift has the similar exponential form under the skew symmetry condition, rather than the constant drift vector $\bar{b}$, in the positive orthant which contains non-smooth part. Moreover, we study the ergodic property of $n-$dimensional system $X(\cdot)$ defined by the SDEs with piece-wise constant coëfficients in Chapter 3, applying those results.

2.4.2 Positive Orthant

In this section we discuss the stationary distribution of $(n-1)-$dimensional reflected Brownian motion $Y(\cdot)$ in the positive orthant for $n \geq 2$. To define the process $Y(\cdot)$ we use the same notation as in Section 2.3.3 and we borrow some of definition from Section 2.4.1. Recall that the $(n-1)-$dimensional Brownian motion $Y(\cdot)$ with data $(\mathfrak{N}, \mathfrak{P}, \mathfrak{Q}, I, \bar{b})$ is well defined through submartingale problem in Proposition 2.4. Assume that the $((n-1) \times (n-1))$ matrices $\mathfrak{N}$ of normal vectors $\mathfrak{n}_i$ and $\mathfrak{Q}$ of tangential vectors $\mathfrak{q}_i$ on the face $\mathfrak{F}_i$ for $i = 1, \ldots, n-1$ satisfies the skew-symmetry condition (2.63) in Remark 2.8. Define the same differential operators (2.95) and their adjoints (2.96) now with $\mathcal{D} \equiv \mathcal{D}_i$ on the ith face $\mathfrak{F}_i$ for $i = 1, \ldots, n-1$ and so on.

We study the basic adjoint relation (2.109). Note that we do *not* have the nice solution $f \in C^{2+\alpha}$ of Dirichlet problem 8 in (2.97). Here instead we start with the basic adjoint relation (2.109) for every $f \in C^2(\bar{\mathfrak{S}})$ and non-negative $\mathfrak{p}(\cdot) \in C^2(\bar{\mathfrak{S}})$, and consider its consequence and relation to the probability density function $\mathfrak{p}(\cdot)$. By Green's theorem and divergence theorem, the basic adjoint relation implies

$$\begin{aligned} 0 &= \int_{\mathfrak{S}} [\mathcal{A}_1 f(y)]\, \mathfrak{p}(y)\, d\,y \\ &= \int_{\mathfrak{S}} f(y) \widehat{\mathcal{A}}_1 \mathfrak{p}(y)\, d\,y + \frac{1}{2} \sum_{i=1}^{n-1} \int_{\mathfrak{F}_i^o} \Big(f\, \frac{\partial \mathfrak{p}}{\partial \mathfrak{n}_i} - \mathfrak{p}\, \frac{\partial f}{\partial \mathfrak{n}_i} - 2 \langle \bar{b}\,, \mathfrak{n}_i \rangle\, \mathfrak{p}\, f + \mathfrak{p} \langle (\mathfrak{n}_i + \mathfrak{q}_i)\,, \nabla f \rangle \Big) (y)\, v(\,d\,y\,) \end{aligned}$$

for every $f \in C^2(\bar{\mathfrak{S}})$. By taking only $f \in C^2$ having compact support in $\mathfrak{S}$, the second term of the right hand is zero, we obtain

$$\widehat{\mathcal{A}}_1 \mathfrak{p}(\cdot) = 0\,, \text{ in } \mathfrak{S}\,. \tag{2.119}$$

Substituting this into the above basic adjoint relation, now we get the second term of the right hand is zero. Moreover, the divergence $\nabla \cdot \mathfrak{q}_i$ of constant $\mathfrak{q}_i$ is zero on the boundary $\mathfrak{F}_i$ for $i = 1, \ldots, n-1$. Since $\mathfrak{q}_i$ is parallel to $\mathfrak{F}_i^o$, the divergence $\nabla \cdot (\mathfrak{q}_i \mathfrak{p}(\cdot) f(\cdot))$ is the same as the divergence taken in the $(n-2)-$dimensional $\mathfrak{F}_i^o$. Then, for every $f \in C_0^2(\bar{\mathfrak{S}})$ we obtain

$$\sum_{i=1}^{n-1} \Big(\int_{\mathfrak{F}_i^o} f(y) (\widehat{\mathcal{D}}_i\, \mathfrak{p}(y) - 2 \langle \bar{b}\,,\, \mathfrak{n}_i \rangle\, \mathfrak{p}(y))\, v(\,d\,y\,) - \sum_{j \neq i} \int_{\mathfrak{F}_{ij}} \langle \mathfrak{q}_i\,, \mathfrak{n}_{ij} \rangle\, \mathfrak{p}(y)\, f(y)\, \widetilde{v}(\,d\,y\,) \Big) = 0\,, \tag{2.120}$$

where $\widetilde{v}(\cdot)$ is the $(n-3)-$dimensional surface measure on each $(n-3)-$dimensional surface $\mathfrak{F}_{ij}$ and $\mathfrak{n}_{ij}$ is the unit vector which is normal to both $\mathfrak{F}_{ij}$ and $\mathfrak{n}_i$, and point into $\mathfrak{F}_i^o$ from $\mathfrak{F}_{ij}$ for $1 \leq i, j \leq n-1$. Such vectors can be written as

$$\mathfrak{n}_{ij} := \frac{\mathfrak{n}_j - \langle \mathfrak{n}_i\,, \mathfrak{n}_j \rangle \mathfrak{n}_i}{[1 - (\langle \mathfrak{n}_i\,, \mathfrak{n}_j \rangle)^2]^{1/2}}\,, \quad \mathfrak{n}_{ji} := \frac{\mathfrak{n}_i - \langle \mathfrak{n}_i\,, \mathfrak{n}_j \rangle \mathfrak{n}_j}{[1 - (\langle \mathfrak{n}_i\,, \mathfrak{n}_j \rangle)^2]^{1/2}}\,. \tag{2.121}$$

By taking $f \in C_0^2$ such that $f(\cdot)|_{\partial \mathfrak{S}}$ has a compact support in $\mathfrak{F}_i^0$, we obtain for each $i = 1, \ldots, n-1$,

$$\widehat{\mathcal{D}}_i\, \mathfrak{p}(\cdot) - 2 \langle \bar{b}\,,\, \mathfrak{n}_i \rangle\, \mathfrak{p}(\cdot) = 0 \;\text{ on } \mathfrak{F}_i^o\,. \tag{2.122}$$

Then, substituting this back into (2.120), we obtain

$$\sum_{i=1}^{n-1} \sum_{1 \leq j \leq i-1} \int_{\mathfrak{F}_{ij}} (\langle \mathfrak{q}_i\,,\, \mathfrak{n}_{ij} \rangle + \langle \mathfrak{q}_j\,,\, \mathfrak{n}_{ji} \rangle)\, \mathfrak{p}(y)\, f(y)\, \widetilde{v}(\,d\,y\,) = 0\,; \quad f \in C_0^2(\bar{\mathfrak{S}})\,. \tag{2.123}$$

Moreover, by taking $f(\cdot) \in C_0^2(\bar{\mathfrak{S}})$ such that its support intersects at most one of the $(n -$

3)−dimensional F_{ij} , we conclude that

$$\langle \mathfrak{q}_i \, , \, \mathfrak{n}_{ij} \rangle + \langle \mathfrak{q}_j \, , \, \mathfrak{n}_{ji} \rangle = 0 \, , \tag{2.124}$$

for $(n-3)-$dimensional F_{ij} . With (2.121) and $\langle \mathfrak{n}_j \, \mathfrak{q}_j \rangle = 0$ for $j = 1, \ldots, n-1$, we conclude that (2.124) is equivalent to (2.63). Thus, we obtain the following Lemma 2.15.

Lemma 2.15 (Harrison & Williams [20]). *The non-negative $C^2(\bar{\mathfrak{S}})-$ function $\mathfrak{p}(\cdot)$ satisfies* (2.109) *for every $f(\cdot) \in C^2(\bar{\mathfrak{S}})$ if and only if the following hold.*

- $\widehat{\mathcal{A}}_1 \, \mathfrak{p}(\cdot) = 0$ *in* $\mathfrak{S}$.
- $\widehat{\mathcal{D}}_i \, \mathfrak{p}(\cdot) = 2\langle \bar{b} \, , \, \mathfrak{n}_i \rangle \, \mathfrak{p}(\cdot)$ *on* $\mathfrak{F}_i$ *for* $i = 1, \ldots n-1$.
- $\mathfrak{n}_i' \mathfrak{q}_j + \mathfrak{n}_j' \mathfrak{q}_i = 0$ *whenever* $\mathfrak{F}_{ij}$ *is* $(n-3)-$*dimensional.*

By the same discussion as in the proof of Proposition 2.5, we obtain the following Proposition 2.6 for the positive orthant $\mathfrak{S}$.

Proposition 2.6 (Harrison & Williams [20]). *Given the data* $(\mathfrak{N}, \mathfrak{P}, \mathfrak{Q}, I, \bar{b})$ *, the* $(n-1)-$*dimensional Brownian motion with reflection in* $\mathfrak{S}$ *has the following equivalent characterizations:*

- *The stationary distribution of the reflected Brownian motion has the exponential form density* $\mathfrak{p}(y) = c \exp(\langle \bar{\gamma} \, , y \rangle)$ *for* $y \in \mathfrak{S}$ *as in (2.117), where* $\bar{\gamma} \in \mathbb{R}^{n-1}$ *and* $c > 0$ *are given by* (2.118) *and*

$$\bar{\gamma} := 2\big(I - \mathfrak{N}^{-1} \, \mathfrak{Q}^{-1}\big)^{-1} \, \bar{b} \, . \tag{2.125}$$

- *The reflection vector field satisfies the skew-symmetry condition* (2.63).

Piecewise constant coëfficients

Suppose that the positive orthant $\mathfrak{S}$ is divided into a set $\{ \mathfrak{S}_\ell \, , \, 1 \le \ell \le m \}$ of finite disjoint regions for some $m \in \mathbb{N}$, i.e., $\mathfrak{S} = \cup_{\ell=1}^m \mathfrak{S}_\ell$ with $\mathfrak{S}_\ell \cap \mathfrak{S}_{\ell'} = \emptyset$ for $\ell \neq \ell'$. We define a piecewise constant function

$$b(\cdot) := \sum_{\ell=1}^{m} \bar{b}_\ell \, \mathbf{1}_{\mathfrak{S}_\ell}(\cdot) \, , \tag{2.126}$$

where $\{\bar{b}_\ell\,,\,\ell = 1,\ldots,n-1\}$ are $(n-1)-$dimensional vectors.

Now we extend the above reasoning to the Brownian motion with reflection for data $(\,\mathfrak{N}\,,\mathfrak{P}\,,\mathfrak{Q}\,,I\,,b(\cdot)\,)$. Let us define the differential operator corresponding to the drift coëfficient $b(\cdot)$ by $\mathcal{A}_2$

$$\mathcal{A}_2\,\varphi(y) := \frac{1}{2}\Delta\varphi(y) + \langle b(y)\,,\nabla\varphi\rangle\,; \quad \varphi \in C^2(\mathbb{R}^n)\,. \tag{2.127}$$

The basic adjoint relation in (2.109) now becomes

$$\int_{\mathfrak{S}} [\mathcal{A}_2\, f(y)]\mathfrak{p}(y)\,d\,y + \frac{1}{2}\int_{\partial\mathfrak{S}} \mathfrak{p}(y)\,\mathcal{D}^1\,f(y)\,v(\,d\,y\,) = 0\,; \quad f \in C^2_0(\bar{\mathfrak{S}})\,, \tag{2.128}$$

which we shall derive in Section 2.4.5. Given this basic adjoint relation we obtain, by applying Green's theorem and the divergence theorem for each region $\mathfrak{S}_\ell$,

$$\begin{aligned}
0 &= \int_{\mathfrak{S}} [\mathcal{A}_2\, f(y)]\,\mathfrak{p}(y)\,d\,y = \int_{\mathfrak{S}} \Big(\frac{1}{2}\Delta\, f(y) + \langle b(y)\,,\nabla\, f(y)\rangle\Big)\mathfrak{p}(y)\,d\,y \\
&= \sum_{\ell=1}^{m} \int_{\mathfrak{S}_\ell} \Big(\frac{1}{2}\Delta\, f(y) + \langle \bar{b}_\ell\,,\nabla\, f(y)\rangle\Big)\mathfrak{p}(y)\,d\,y \\
&= \int_{\mathfrak{S}} f(y)\widehat{\mathcal{A}}_2\mathfrak{p}(y)\,d\,y \\
&\quad + \frac{1}{2}\int_{\mathfrak{F}_i^\circ} \Big(f\,\frac{\partial\mathfrak{p}}{\partial\mathfrak{n}_i} - \mathfrak{p}\,\frac{\partial f}{\partial\mathfrak{n}_i} - 2\langle b(y)\,,\mathfrak{n}_i\rangle\,\mathfrak{p}\,f + \mathfrak{p}\langle(\mathfrak{n}_i + \mathfrak{q}_i)\,,\nabla f\rangle\Big)(y)\,v(\,d\,y\,)\,.
\end{aligned}$$

We go through (2.119) to (2.124) to obtain the following Proposition 2.7.

Proposition 2.7. *Given the data* $(\,\mathfrak{N}\,,\mathfrak{P}\,,\mathfrak{Q}\,,I\,,b(\cdot)\,)$ *, the* $(n-1)-$*dimensional Brownian motion with reflection in* $\mathfrak{S}$ *has the following equivalent characterization.*

- *The stationary distribution of the reflected Brownian motion has the exponential form density* $\mathfrak{p}(y) = c\,\exp(\langle\gamma(y)\,,y\rangle)$ *for* $y \in \mathfrak{S}$ *as in (2.117), where* $\bar{\gamma} \in \mathbb{R}^{n-1}$ *and* $c > 0$ *are given by*

$$\gamma(\cdot) := 2(I - \mathfrak{N}^{-1}\,\mathfrak{Q})^{-1}b(\cdot) \;\; \textit{in}\ \mathfrak{S}\,, \quad c := \Big(\int_{\mathfrak{S}} \exp\big(\langle\gamma(y)\,,\,y\,\rangle\big)\,d\,y\Big)^{-1}. \tag{2.129}$$

- *The reflection vector field satisfies the skew-symmetry condition* (2.63).

In the following we shall see that the stationary distribution exists and is unique, and obtain the results for the reflected Brownian motion with data $(\,\mathfrak{N}\,,\mathfrak{P}\,,\mathfrak{Q}\,,A\,,b(\cdot)\,)$, where A is an $((n-1)\times(n-1))$ positive definite symmetric matrix.

2.4.3 Existence of Stationary Distribution

In the previous Section 2.4.2 we studied the relationship between the stationary distribution and the skew-symmetry condition. In this section we show the existence of stationary distribution of Brownian motion $Y(\cdot)$ with reflection defined in (2.47) and (2.59) for the data $(\mathfrak{N}, \mathfrak{P}, \mathfrak{Q}, A, b(\cdot))$ in the positive orthant $\mathfrak{S}$. Here in this section $A = \sigma\sigma'$ is the $((n-1) \times (n-1))$ positive definite symmetric matrix and $b(\cdot)$ is a piecewise constant function defined in (2.126): the process $Y(\cdot)$ has the decomposition

$$Y(t) = Y(0) + \xi(t) + (I - \mathfrak{P})\,\Lambda(t) = Y(0) + \xi(t) + \mathfrak{R}\,\Lambda(t) \in \mathfrak{S}\,; \quad 0 \le t < \infty\,.$$

where the pair $(\Lambda(\cdot), Y(\cdot))$ has the representation (2.48) and $\xi(\cdot)$ is in (2.59). The $((n-1) \times (n-1))$ matrix $\mathfrak{R} := (I - \mathfrak{P})$ is the reflection matrix which represents the direction of reflection.

By the submartingale characterization, $\Lambda(\cdot)$ is minimal in the sense that for any $(n-1)-$dimensional process $V(\cdot)$ of bounded variation such that $\xi(\cdot) + \mathfrak{R}\,V(\cdot) \in \mathfrak{S}$, the non-decreasing process $\Lambda_i(\cdot)$ is smaller than or equal to $V_i(\cdot)$ for $i = 1, \ldots, n-1$. Let us denote by $\mathfrak{C}$ the class of $(n-1)-$dimensional process $V(\cdot)$ of bounded variation such that $\xi(t) + \mathfrak{R}\,V(t) \in \mathfrak{S}$ for $0 \le t < \infty$.

For the sake of simplicity of reasoning, let us introduce the linear transformation of rotation and scaling. Define $Y^*(\cdot) := \mathfrak{R}^{-1} Y(\cdot)$ and $\xi^*(\cdot) := \mathfrak{R}^{-1}\,\xi(\cdot)$. Similarly, we transform the state space $\mathfrak{S}$ to $\mathfrak{S}^* := \mathfrak{R}^{-1}\,\mathfrak{S}$. Let us denote the probability distribution induced from $\xi^*(\cdot)$ starting at $\xi(0) = 0$ by $\mathbb{P}_0^*$.

The following Lemma 2.16 is essential for existence of stationary distribution.

Lemma 2.16. *For $y^* \in \mathfrak{S}^*$ we define the limit distribution $F^*(y^*) := \lim_{t\to\infty} \mathbb{P}_0(Y^*(t) \le y^*)$, where the inequality is evaluated element-wise. Then, the probability $\mathbb{P}_0(Y^*(t) \le y^*)$ decreases monotonically in y^*, and*

$$F^*(y^*) = \mathbb{P}_0^*(\exists V \in \mathfrak{C}\,, y^* - \xi^*(t) - V(t) \in \mathfrak{S}^*\ , 0 \le t < \infty)\,. \tag{2.130}$$

Moreover, if the piecewise constant drift coëfficient $b(\cdot)$ in (2.126) and the reflection matrix $\mathfrak{R}$ satisfies $\mathfrak{R}^{-1}\,b(\cdot) < 0$ for each element, then there exists $\delta \in \mathfrak{S}$ such that

$$F^*(y^*) \ge 1 - \sum_{i=1}^{n-1} \exp\big(-2\delta'\,\mathfrak{D}^{-1}\,y\big)\,, \tag{2.131}$$

where $y = \mathfrak{R}\, y^*$ *and* $\mathfrak{D} =$ *diag* (A) .

Harrison & Williams [19] showed this result for constant vector $b(\cdot) \equiv \bar{b}$. We adapt their proof and show Lemma 2.16 for piecewise constant $b(\cdot)$.

Proof. Fix $t \in [0, \infty)$ and define a process $\check{\xi}(s) := (\xi(t) - \xi(t-s))\, \mathbf{1}_{\{0 \le s \le t\}} + (\xi(s) - \xi(0))\mathbf{1}_{\{t \le s < \infty\}}$ before and after $\xi(t)$, and other related processes $\check{\Lambda}(\cdot) := \Psi(\check{\xi}(\cdot))$, $\check{Y}(\cdot) := (I + \Psi)(\check{\xi}(\cdot))$, $\check{Y}^* := \mathfrak{R}^{-1}\check{Y}(\cdot)$ and $\check{\xi}^* := \mathfrak{R}^{-1}\check{\xi}(\cdot)$. Note that when the starting point is zero, $\mathbb{P}_0 \equiv \mathbb{P}_0^*$. So, we consider (2.130) for $\check{Y}^*$ in place of Y^*. We observe $\mathbb{P}_0^*$-a.s. that

$$
\begin{aligned}
\{\, \omega : \check{Y}^*(t) \le y^* \,\} &= \{\omega :\, \exists V \in \mathfrak{C}\,, \check{\xi}^*(s) + V(s) \in \mathfrak{S}^*\,, 0 \le s \le t\,; \check{\xi}^*(t) + V(t) \le y^* \,\} \\
&= \{\omega :\, \exists V \in \mathfrak{C}\,, \check{\xi}^*(s) + V(s) \in \mathfrak{S}^*\,, 0 \le s \le t\,; \check{\xi}^*(t) + V(t) = y^* \,\} \\
&= \{\omega :\, \exists V \in \mathfrak{C}\,, \xi^*(t) - \xi^*(t-s) + V(s) - V(t-s) \in \mathfrak{S}^*\,, \\
&\qquad\qquad\qquad 0 \le s \le t\,; \check{\xi}^*(t) + V(t) = y^* \,\} \\
&= \{\omega :\, \exists V \in \mathfrak{C}\,, \xi^*(t) - \xi^*(s) + V(t) - V(s) \in \mathfrak{S}^*\,, \\
&\qquad\qquad\qquad 0 \le s \le t\,; \check{\xi}^*(t) + V(t) = y^* \,\} \\
&= \{\omega :\, \exists V \in \mathfrak{C}\,, y^* - \xi^*(s) - V(s) \in \mathfrak{S}^*\,, 0 \le s \le t\,; \check{\xi}^*(t) + V(t) = y^* \,\} \\
&= \{\omega :\, \exists V \in \mathfrak{C}\,, y^* - \xi^*(s) - V(s) \in \mathfrak{S}^*\,, 0 \le s \le t \,\}\,.
\end{aligned}
$$

The last set decreases monotonically to the set in the right-hand of (2.130).

Since the piecewise constant drift $b(\cdot)$ in (2.126) satisfies $\mathfrak{R}^{-1}\, b(\cdot) < 0$ for each element, i.e., $\mathfrak{R}^{-1}\bar{b}_\ell < 0$ for each element and $\ell = 1, \ldots, m$, there exists a sufficiently small $\delta \in \mathfrak{S} = (\mathbb{R}_+)^{n-1}$ such that $\mathfrak{R}^{-1}\, \delta \le -\mathfrak{R}^{-1}\, \bar{b}_\ell$ for $\ell = 1, \ldots, m$. Then, define $\bar{c} := \min_{1 \le \ell \le m} \mathfrak{R}^{-1}\, (-\bar{b}_\ell - \delta) \in \mathfrak{S}$ where the minimum is taken for each element. Take the continuous process $V(t) := \bar{c}t$ which is in $\mathfrak{C}$. It follows from (2.130) that

$$
\begin{aligned}
F^*(y^*) &\ge \mathbb{P}_0^*(y^* - \xi^*(t) - \bar{c}t \in \mathfrak{S}^*\,,\ 0 \le t < \infty) \\
&= \mathbb{P}_0(y - \xi(t) - (\mathfrak{R}\bar{c})t \in \mathfrak{S}\,,\ 0 \le t < \infty) \\
&= \mathbb{P}_0\Big(y - \sigma B(t) - \int_0^t \big(\sum_{\ell=1}^m \bar{b}_\ell \mathbf{1}_{\mathfrak{S}_\ell}(\xi(s)) + \mathfrak{R}\bar{c}\big) d\, s \in \mathfrak{S}\,,\ 0 \le t < \infty\Big)\,.
\end{aligned}
$$

Since $-(\bar{b}_\ell + \mathfrak{R}\bar{c}) \ge \delta$ for each element and $\ell = 1, \ldots, m$ and $\mathfrak{S} = (\mathbb{R}_+)^{n-1} = \cup_{\ell=1}^m \mathfrak{S}_\ell$, we obtain

$$
\begin{aligned}
F^*(y^*) &\geq \mathbb{P}_0(\cap_{i=1}^{n-1}\{y_i + W_i(t) + \delta_i t \geq 0\,,\ \ 0 \leq t < \infty\}) \\
&\geq 1 - \sum_{i=1}^{n-1} \mathbb{P}_0(y_i + \inf_{t\geq 0}(W_i(t) + \delta_i t) < 0) = 1 - \sum_{i=1}^{n-1} \exp(-2\,\delta' \mathfrak{D}^{-1}\, y)\,,
\end{aligned} \tag{2.132}
$$

where $W(\cdot) := -\sigma B(\cdot)$ is the $(n-1)-$dimensional standard Brownian motion with variance-covariance rate A starting at the origin, and $\mathfrak{D} =$ diag (A). This is (2.131) and completes the proof of Lemma 2.16. □

It follows from (2.131) that the limit distribution is non-trivial if $\mathfrak{R}^{-1}\, b(\cdot) < 0$ element-wise. Let us denote by ν^* this limit distribution on $\mathfrak{S}^*$. By transforming back to $\mathfrak{S}$, we define the probability measure $\nu(A) := \nu^*(\mathfrak{R}^{-1}A)$ for all $A \in \mathcal{B}(\mathfrak{S})$. Thus, for $f \in C_b(\mathfrak{S})$ by Markov property we obtain

$$
\begin{aligned}
\int_{\mathfrak{S}} d\,\nu(y)\, f(y) &= \lim_{s\to\infty} \mathbb{E}_0[f(Y(s+t))] = \lim_{s\to\infty} \mathbb{E}_0[\,\mathbb{E}_{Y(s)}[\,f(Y(t))\,]\,] \\
&= \int_{\mathfrak{S}} d\,\nu(y) \mathbb{E}_y[\,f(Y(t))\,]\,,
\end{aligned}
$$

which leads us to the conclusion that $\pi(\cdot)$ is a stationary distribution of $Y(\cdot)$ through a monotone class argument. Thus, we obtain the following Proposition 2.8.

Proposition 2.8. *If $\mathfrak{R}^{-1}b(\cdot) < 0$ holds element-wise in the $(n-1)-$dimensional positive orthant $\mathfrak{S}$, then the reflected Brownian motion $Y(\cdot)$ in $\mathfrak{S}$ with data $(\,\mathfrak{N}\,, \mathfrak{P}\,, \mathfrak{Q}\,, A\,, b(\cdot)\,)$ has a stationary distribution.*

2.4.4 Uniqueness of Stationary Distribution

In this section we see that the distribution of process $Y(\cdot)$ is mutually absolutely continuous with respect to Lebesgue measure $Leb(\cdot)$. Then, if $Y(\cdot)$ has a stationary distribution $\nu(\cdot)$, it is also mutually absolutely continuous with respect to Lebesgue measure $Leb(\cdot)$. Since $Y(\cdot)$ is Markov process, by (individual) ergodic theorem, for any bounded continuous function $f : \mathbb{R}^{n-1} \to \mathbb{R}$,

$$
\lim_{T\to\infty} \frac{1}{T} \sum_{t=1}^{T} \mathbb{E}_y\left[f(Y(t))\right] = \int_{\mathfrak{S}} f(z) d\,\nu\,(z) \qquad \nu a.s. - y\,, \tag{2.133}
$$

and hence y $Leb - a.e.$. This implies the stationary distribution $\nu(\cdot)$ is uniquely determined as the limit. So, it is sufficient to show the following lemma.

Lemma 2.17 ([19])**.** *For each $x \in \mathbb{R}^n$, $\pi \in \Pi$, we have*

$$\mathbb{E}_x \Big[\int_0^\infty 1_{\partial \mathfrak{S}}(X(s)) ds \Big] = 0 \,, \tag{2.134}$$

where $\partial \mathfrak{S}$ is the boundary of $\mathfrak{S}$.

Using this lemma and the strong Markov property of the continuous process $Y(\cdot)$, we obtain the absolute continuity of the distribution of $Y(\cdot)$ with respect to Lebesgue measure through Fubini's theorem.

Lemma 2.18 ([19])**.** *$\mathbb{P}_y(\cdot)$ and $Leb(\cdot)$ are absolutely continuous with respect to each other.*

Thus, as a summary of the above argument we obtain the following result

Proposition 2.9. *If the piecewise constant drift $b(\cdot)$ in* (2.126) *satisfies $\mathfrak{R}^{-1} b(\cdot) < 0$ for each element, then the $(n-1)-$dimensional Brownian motion $Y(\cdot)$ with reflection and the data $(\,\mathfrak{N}\,, \mathfrak{P}\,, \mathfrak{Q}\,, A\,, b(\cdot)\,)$, has a unique stationary distribution.*

2.4.5 Properties of Additive Functionals

Representation of additive functionals

In this section we review representation theorem for additive functional. This comes from the strong Markov property of X and Kingman's sub-additive ergodic theorem [32]. A measurable, adapted, real-valued process $A = \{A_t, \mathcal{F}_t; t \geq 0\}$ is called an additive functional if $A_{t+u}(\omega) = A_t(\omega) + A_u(\theta_u \omega)$ for $0 \leq s, t \leq \infty$, $\mathbb{P} - a.e. \omega \in \Omega\,,\ x \in \mathbb{R}^n$, where $\theta.$ is the shift operator, i.e., $\theta_s\, \omega(t) = \omega(s+t)$ for $0 \leq s\,, t < \infty$. A typical examples of additive functional is the local time. One can show that the non-decreasing process $\Lambda_i(\cdot)$ in (2.47) has a modification which is an additive functional for $i = 1, \ldots, n-1$ [19].

As in the previous Section 2.4.4, consider that $Y(\cdot)$ has the stationary distribution $\nu(\cdot)$, i.e.,

$$\nu(B) = \int_{\mathfrak{S}} \mathbb{P}_y(Y(t) \in B)\, \nu(d\,y) \,; \quad B \in \mathcal{B}(\mathbb{R}^{n-1}) \,.$$

For a bounded non-negative measurable function $f(\cdot)$ and the non-decreasing part $\Lambda_i(\cdot)$ in the representation of $Y(\cdot)$ in (2.48), define the functional $\mathfrak{m}_i(f)$ associated with the additive functional Λ_i by

$$\begin{aligned} \mathfrak{m}_i(f) &:= \sup_{t>0} \frac{1}{t} \int_{\mathfrak{S}} \mathbb{E}_y \Big[\int_0^t f(Y_i(s))\, d\,\Lambda_i(s) \Big]\, d\,\nu(\,y) \\ &= \sup_{t>0} \frac{1}{t} \mathbb{E}_\nu \Big[\int_0^t f(Y_i(s))\, d\,\Lambda_i(s) \Big] =: \sup_{t>0} \frac{1}{t}\, \mathbb{E}_\nu [\, S_i(t) \,]\; ; \quad i = 1, \ldots, n-1\, , \end{aligned} \tag{2.135}$$

where $S_i(\cdot)$ defined by $\int_0^\cdot f(Y_i(s))\, d\,\Lambda_i(s)$ and $\mathbb{E}_\nu$ is the expectation with respect to the probability distribution $\mathbb{P}_\nu$ induced from $Y(\cdot)$ with the initial probability distribution being the stationary distribution $\nu(\cdot)$. Note that $\Lambda_i(\cdot)$ is an adapted non-decareasing process which increases only at the time t such that $\{\, t \ge 0 \,:\, Y_i(t) \,\}$ for $i = 1, \ldots, n-1$. Thus, $S_i(\cdot)$ is the average of $f(Y_i(\cdot)$ with respect to the random measure $\Lambda_i(\cdot)$ of how much the process $Y_i(\cdot)$ stays in the neighborhood of the origin. This $S_i(\cdot)$ is additive, i.e.,

$$\begin{aligned} S_i(t,\omega) + S_i(u, \theta_t\, \omega) &= \int_0^t f(Y_i(s,\omega))\, d\,\Lambda_i(s,\omega) + \int_0^u f(Y_i(s, \theta_t\, \omega))\, d\,\Lambda_i(s, \theta_t\, \omega) \\ &= \int_0^t f(Y_i(s,\omega))\, d\,\Lambda_i(s) + \int_t^{u+t} f(Y_i(s,\omega))\, d\,\Lambda_i(s,\omega) \Big] \\ &= S_i(t+u, \omega)\, ; \quad 0 \le t\, , u < \infty\, , \end{aligned}$$

by the Markov property. Then, by Kingman's sub-additive theorem [32],

$$\mathfrak{m}_i(f) = \lim_{t \to 0} \frac{1}{t} \mathbb{E}_\nu \Big[\int_0^t f(Y_k)\, d\,\Lambda_i(s) \Big]\, ; \quad i = 1, \ldots n-1\, .$$

Especially, since $\nu(\cdot)$ is invariant, we can write

$$\mathfrak{m}_i(f) = \mathbb{E}_\nu [\int_0^1 f(Y_i(s))\, d\,\Lambda_i(s) \,]\, ; \quad i = 1, \ldots, n-1\, . \tag{2.136}$$

Since the behavior of $Y(\cdot)$ in $\overset{o}{\mathfrak{S}}$ can be seen as Brownian motion with bounded drift and volatility, there is a constant $C > 0$ such that for $i = 1, \ldots, n{-}1$, and $t \ge 0$, $\mathbb{E}_y[\Lambda_i(t)] \le C(t{+}1)$. Hence, the measure $\mathfrak{m}_i(f)$ associated with $\Lambda_i(\cdot)$ is finite. Moreover, the functional $\mathfrak{m}_i(f)$ is mutually absolutely continuous with respect to $(n-2)-$dimensional Lebesgue measure on $\mathfrak{F}_i$, i.e.,

$$\mathfrak{m}_i(f) = \int_{\mathfrak{F}_i} f(y)\, v(\, d\, y\,)\, , \tag{2.137}$$

where $v(\cdot)$ is the $(n{-}2)-$dimensional Lebesgue measure on each i-th face $\mathfrak{F}_i$ for $i = 1, \ldots, n{-}1$.

Now we are ready to state the main result. The result is a generalization of Harrison & Williams [20] from the constant drift coëfficients $b(\cdot) \equiv \bar{b}$ to the piecewise constant drift coëfficients $b(\cdot)$ defined in (2.126).

2.4.6 Main result

Derivation of basic adjoint relation

By Itô's formula and the submartingale characteristics, we obtain that

$$f(Y(t)) - f(Y(0)) - \int_0^t \sum_{i=1}^{n-1} \mathcal{D}_i f(Y(s))\, d\,\Lambda_i - \int_0^t \mathcal{A}_2\, f(Y(s))\, d\,s\,;\;\; 0 \le t < \infty \tag{2.138}$$

is $\mathbb{P}_y-$martingale, where $\mathcal{A}_2$ is defined in (2.127) for $f \in C_b^2(\mathfrak{S})$. Taking expectations under $\mathbb{P}_\nu$ and combining with (2.136), we obtain

$$t \int_{\mathfrak{S}} \mathcal{A}_2 f(y)\, \nu(\,d\,y\,) + t \sum_{i=1}^{n-1} \int_{\mathfrak{F}_i} \mathcal{D}_i\, f(y)\, v(\,d\,y\,) = 0\,; \quad t \ge 0.$$

Since the stationary distribution has a density $\mathfrak{p}(\cdot)$, it becomes the basic adjoint relation (2.128). Thus, we obtain the following

Proposition 2.10. *If $\nu(\cdot)$ is the stationary distribution absolutely continuous with respect to Lebesgue measure, then for each $f \in C_b^2(\mathfrak{S})$, the probability density $\mathfrak{p}(\cdot)$ satisfies the basic adjoint relation* (2.128).

Now with Propositions 2.7, 2.8, 2.9 and 2.10 we are ready to state the following main result in Section 2.4.

Theorem 2.6. *Consider the $(n-1)-$dimensional reflected Brownian motion $Y(\cdot)$ with data $(\,\mathfrak{N}\,, \mathfrak{P}\,, \mathfrak{Q}\,, A\,, b(\cdot))$, where $b(\cdot)$ is piecewise constant defined in* (2.126)*. If the stability condition $\mathfrak{R}^{-1}\, b(\cdot) < 0$ is satisfied element-wise in $\mathfrak{S}$, and the skew-symmetry condition* (2.60) *holds, then the stationary distribution with the density $\mathfrak{p}(\cdot)$ has product form which can be written as*

$$\mathfrak{p}(y) = c \cdot \exp\big(\, - 2\langle\; diag\;(\mathfrak{R})[\; diag\;(A)\,]^{-1} \mathfrak{R}^{-1}\, b(y)\,,\, y\,\rangle\;\big)\;: \quad y \in \mathfrak{S}\,. \tag{2.139}$$

Here c is the normalizing constant such that $\int_{\mathfrak{S}} \mathfrak{p}(y)\, d\, y = 1$; and diag$(\mathfrak{R})$, diag A are $((n-1)\times(n-1))$ diagonal matrix whose diagonal elements are the same as $\mathfrak{R}$ and A, respectively.

Remark 2.12. Conversely, one can show that if the stationary distribution has the product form (2.139), then the stability condition $\mathfrak{R}^{-1} b(\cdot) < 0$ element-wise and the skew-symmetry condition hold. See [19]. When $A \equiv I$ and $b(\cdot) \equiv \bar{b}$, the form of density function is consistent with the previous results (2.125) and (2.129) in Proposition 2.6 and 2.7 obtained by Williams [61].

Chapter 3

Rankings and Ergodicity

In this chapter we utilize the results in the previous sections to study rankings, attainability, and ergodic behavior of the $n-$dimensional process $X(\cdot)$ defined in (1.4) with piecewise constant coëfficient (1.7) in Chapter 1. Recall that through the martingale problem and some estimates of Alexandrov's type we see the solution to the stochastic differential equation with bounded coëfficients $b(\cdot)$ and $s(\cdot)$ in (1.4):

$$d\,X(t) = b(X(t))\,d\,t + s(X(t))\,dW(t)\,; \quad 0 \le t < \infty\,, \quad X(0) = x_0 \in \mathbb{R}^n \tag{3.1}$$

exists. The uniqueness of probability distribution of solution is verified, when the diffusion coëfficient $s(\cdot)$ satisfies (1.7):

$$b(x) := \sum_{\pi \in \Pi} b_\pi \mathbf{1}_{\mathcal{R}_\pi}(x) = b_{\mathfrak{p}^x}\,, \quad s(x) := \sum_{\pi \in \Pi} s_\pi \mathbf{1}_{\mathcal{R}_\pi}(x) = s_{\mathfrak{p}^x}\,; \quad x \in \mathbb{R}^n \tag{3.2}$$

for polyhedral region $\mathcal{R}_\pi\,, \pi \in \Pi$ of (1.2).

3.1 Rankings of Multidimensional Diffusion

Given an $(n \times 1)$ vector process $X(\cdot) := \{(X_1(t), \ldots, X_n(t)\,; 0 \le t < \infty\}$, we define the vector $X_{(\cdot)} := \{(X_{(1)}(t), \ldots, X_{(n)}(t))\,; 0 \le t < \infty\}$ of *ranked processes,* ordered from largest to smallest, by

$$X_{(k)}(t) := \max_{1 \le i_1 < \ldots < i_k \le n} \Big(\min \big(X_{i_1}(t), \ldots, X_{i_k}(t) \big) \Big)\,; \quad 0 \le t < \infty\,, \quad k = 1, \cdots, n\,, \tag{3.3}$$

where the ties are resolve in (3.6). By the definition, we have the *rankings* of multidimensional diffusion:

$$X_{(1)}(t) \geq X_{(2)}(t) \geq \cdots \geq X_{(n)}(t)\,; \quad 0 \leq t < \infty\,.$$

Problem 9. *What is the dynamics of ranked process* $X_{(\cdot)}(\cdot)$ *? What are the properties?*

To answer this question let us start with $n = 2$. Recall that the local time $L(\cdot)$ of one-dimensional process $Y(\cdot)$ at level zero is defined as

$$2\,L(t) = Y(t) - Y(0) - \int_0^t \operatorname{sgn}(Y(s))\,dY(s)\,; \quad 0 \leq t < \infty\,.$$

By Tanaka-Meyer's formula of local times, we have

$$\begin{aligned}(X_1(t) - X_2(t))^+ &= (X_1(0) - X_2(0))^+ + \int_0^t \mathbf{1}_{(0,\infty)}(X_1(s) - X_2(s))\,d(X_1 - X_2)(s)\\ &\qquad + L^{X_1 - X_2}(t)\,,\\ (X_1(t) - X_2(t))^- &= (X_1(0) - X_2(0))^- + \int_0^t \mathbf{1}_{(-\infty,0]}(X_1(s) - X_2(s))\,d(X_1 - X_2)(s)\\ &\qquad + L^{X_1 - X_2}(t)\,; \quad 0 \leq t < \infty\,,\end{aligned} \tag{3.4}$$

where $L^{X_1 - X_2}(\cdot)$ is the local time of $X_1(\cdot) - X_2(\cdot)$ at level zero. Since $x_{(1)} := \max(x_1\,, x_2) = (x_1 - x_2)^+ + x_2$ and $x_2 := \min(x_1\,, x_2) = -(x_1 - x_2)^- + x_2$, we can write the dynamics of rankings for $n = 2$:

$$d\,X_{(1)}(t) = d\,(X_1(t) - X_2(t))^+ + d\,X_2(t)\,, \quad d\,X_{(2)}(t) = -d\,(X_1(t) - X_2(t))^- + d\,X_2(t)\,;$$

for $0 \leq t < \infty$, where the dynamics of $(\cdot)^{\pm}$ is computed from the above Tanaka-Meyer's formula (3.4).

For general $n \geq 2$ these involve the local times $\Lambda^{k,\ell}(\cdot) \equiv L^{X_{(k)} - X_{(\ell)}}(\cdot)$ for $1 \leq k < \ell \leq n$, where the notation $L^Y(\cdot)$ is used to signify the local time at the origin of a continuous semimartingale $Y(\cdot)$. An increase in $\Lambda^{k,\ell}(\cdot)$ is due to, and signifies, a collision of $\ell - k + 1$ particles of $X(\cdot)$ in the ranks k through ℓ. In general, when multiple collisions can occur, there are $(n-1)\,n/2$ such possible local times; all of them appear then in the dynamics of the ranked processes, in the manner of Banner & Ghomrasni [6].

Let $S_k(t) := \{i\, :\, X_i(t) = X_{(k)}(t)\,\}$ be the set of indexes of processes which are kth ranked, and denote its cardinality by $N_k(t) := |S_k(t)|$ for $0 \leq t < \infty$.

Proposition 3.1 (Theorem 2.3 of Banner & Ghomrasni [6]). *For any $n-$dimensional continuous semi-martingale process* $X(\cdot) = (X_1(\cdot), \ldots, X_n(\cdot))$ *, its ranked process* $X_{(\cdot)}(\cdot)$ *with components* $X_{(k)}(t) = X_{p_t(k)}(t)$*,* $k = 1, \ldots, n$ *can be written as*

$$d\,X_{(k)}(t) = (N_k(t))^{-1}\Big[\sum_{i=1}^{n} \mathbf{1}_{\{X_{(k)}(t)=X_i(t)\}}\,d\,X_i(t) + \sum_{j=k+1}^{n} d\,\Lambda^{k,j}(t) - \sum_{j=1}^{k-1} d\,\Lambda^{j,k}(t)\Big]\,, \tag{3.5}$$

for $0 \le t < \infty$ *. Here* $p_t := \{\,(p_t(1), \ldots, p_t(n))\,\}$ *is the random permutation of* $\{1, \ldots, n\}$ *which describes the relation between the indexes of* $X(t)$ *and the ranks of* $X_{(\cdot)}(t)$ *such that*

$$p_t(k) < p_t(k+1)\,, \quad \textit{if} \quad X_{(k)}(t) = X_{(k+1)}(t)\,; \quad 0 \le t < \infty\,. \tag{3.6}$$

Recall that Π is the symmetric group of permutations of $\{\,1, \ldots, n\,\}$. The map $p_t : \Omega \times [0, \infty) \mapsto \Pi$ is measurable with respect to σ-field generated by the adapted continuous process $\{X(s)\,, 0 \le s \le t\}$ and hence is predictable. Since Π is bijective, let us define the inverse map $p_t^{-1} := (p_t^{-1}(1), \ldots, p_t^{-1}(n))$ such that

$$X_{(p_t^{-1}(i))}(t) = X_i(t)\,; \quad i = 1, \ldots, n\,, \quad 0 \le t < \infty\,. \tag{3.7}$$

That is, $p_t^{-1}(i)$ indicates the rank of $X_i(t)$ in the $n-$dimensional process $X(t)$. The map $p_t^{-1} : \Omega \times [0, \infty) \mapsto \Pi$ is also predictable.

Let us recall the triple collision problem in Section 1.6:

$$\begin{aligned} &\mathbb{P}_{x_0}(X_i(t) = X_j(t) = X_k(t) \ \text{ for some } t \ge 0\,) = 0\,, \ \text{ or} \\ &\mathbb{P}_{x_0}(X_i(t) = X_j(t) = X_k(t)\ i.o.) = 1\,; \quad x_0 \in \mathbb{R}^n\,, \end{aligned} \tag{3.8}$$

for some $1 \le i \le j \le k \le n$. Under the assumption of "no triple collisions" (that is, when the only non-zero change-of-rank local times are those of the form $\Lambda^{k,k+1}(\cdot)\,,\ 1 \le k \le n-1\,$), Fernholz [12] considered the stochastic differential equation of the vector of ranked process $X_{(\cdot)}$ in a general framework and Banner, Fernholz & Karatzas [5] obtained a rather complete analysis of the Atlas model (3.38).

Let us take a close look at triple collision with the rankings of diffusion. If, for every $j =$

$1, \ldots, n-2$, the two-dimensional process

$$\big(Y_j(\cdot),\, Y_{j+1}(\cdot)\big)' := \big(X_{(j)}(\cdot) - X_{(j+1)}(\cdot),\, X_{(j+1)}(\cdot) - X_{(j+2)}(\cdot)\big)' \tag{3.9}$$

obtained by looking at the "gaps" among the three adjacent ranked processes

$$X_{(j)}(\cdot),\; X_{(j+1)}(\cdot),\; X_{(j+2)}(\cdot)\,,$$

never reaches the corner $(0,0)'$ of $\mathbb{R}^2$ almost surely, then the process $X(\cdot)$ satisfies

$$\mathbb{P}_{x_0}\big(X_i(t) = X_j(t) = X_k(t)\,,\ \text{for some } (i,j,k)\,,\ t > 0\big) = 0 \tag{3.10}$$

for $x_0 \in \mathbb{R}^n \setminus \mathcal{Z}$. On the other hand, if for some $j = 1, \ldots, n-2$ the vector of gaps $(\,X_{(j)}(\cdot) - X_{(j+1)}(\cdot),\, X_{(j+1)}(\cdot) - X_{(j+2)}(\cdot)\,)'$ does reach the corner $(0,0)'$ of $\mathbb{R}^2$ almost surely, then we have

$$\mathbb{P}_{x_0}\big(X_i(t) = X_j(t) = X_k(t),\ \text{for some } (i,j,k),\ t > 0\big) \;=\; 1\,; \quad x_0 \in \mathbb{R}^n.$$

Since the above the two-dimensional differenced process $(\,X_{(j)}(\cdot) - X_{(j+1)}(\cdot),\, X_{(j+1)}(\cdot) - X_{(j+2)}(\cdot)\,)'$ is non-negative and reflects back instantaneously when one of its coördinates attains zero, it looks like the Brownian motion with reflection. Thus, we study the ranked process $X_{(\cdot)}$ and itsadjacent differences as an application of study on multidimensional Brownian motion with reflection of Chapter 2.

3.2 No Triple Collisions

Let us recall a collection $\{Q_k^{(i)}\}_{1 \le i,k \le n}$ of polyhedral domains in $\mathbb{R}^n$, such that $\{Q_k^{(i)}\}_{1 \le i \le n}$ is partition $\mathbb{R}^n$ for each fixed k, and $\{Q_k^{(i)}\}_{1 \le k \le n}$ is partition $\mathbb{R}^n$ for each fixed i. The interpretation is as follows:

$$x = (x_1, \cdots, x_n)' \in Q_k^{(i)} \quad \text{means that } x_i \text{ is ranked } k\text{th among } x_1, \cdots, x_n\,,$$

with ties resolved by resorting to the smallest index for the highest rank, by analogy with (3.3).

Our main observation is the following Proposition 3.2.

Proposition 3.2 (Ichiba & Karatzas [23]). *For $n \ge 3$, consider the weak solution of the equation*

(3.1) *with piece-wise constant diffusion coëfficient, where* $s(\cdot)$ *is the diagonal matrix*

$$s(x) := diag \left(\sum_{k=1}^{n} \tilde{\sigma}_k 1_{Q_k^{(1)}}(x), \ldots, \sum_{k=1}^{n} \tilde{\sigma}_k 1_{Q_k^{(n)}}(x) \right) ; \quad x \in \mathbb{R}^n . \tag{3.11}$$

If the positive constants $\{ \tilde{\sigma}_k \, ; 1 \le k \le n \}$ *satisfy the linear growth condition*

$$\widetilde{\sigma}_2^2 - \widetilde{\sigma}_1^2 \, = \, \widetilde{\sigma}_3^2 - \widetilde{\sigma}_2^2 \, = \, \cdots \, = \, \widetilde{\sigma}_n^2 - \widetilde{\sigma}_{n-1}^2 \, , \tag{3.12}$$

then (3.10) *holds, i.e., there are no triple-collisions among the* n *one-dimensional particles.*

If $n = 3$*, the weaker condition* $\widetilde{\sigma}_2^2 - \widetilde{\sigma}_1^2 \ge \widetilde{\sigma}_3^2 - \widetilde{\sigma}_2^2$ *is sufficient for the absence of triple collisions.*

Remark 3.1. This special structure (3.11) has been studied in the context of mathematical finance. Recent work on interacting particle systems by Pal & Pitman [46] clarifies the long-range behavior of the spacings between the arranged Brownian particles under the equal variance condition: $\widetilde{\sigma}_1 = \cdots = \widetilde{\sigma}_n$. The setting of systems with countably many particle is also studied there, and related work from the Physics literature on competing tagged particle systems is surveyed. □

Remark 3.2. In the above result the drift coëfficients $b(\cdot)$ in (3.1) do not affect the conclusion as in Section 1.6. In fact, we consider Atlas model studied by Fernholz [12] and Banner, Fenholz & Karatzas [5], and its extension, called the *hybrid* Atlas model, in Section 3.3. □

3.2.1 Brownian Motion with Reflection

Recall the notations in Chapter 2. We shall define Brownian motion with reflection on the faces of the non-negative orthant

$$\mathfrak{S} := \mathbb{R}_+^{n-1} = \left\{ \sum_{k=1}^{n-1} x_k e_k \, : \, x_1 \ge 0, \ldots, \, x_{n-1} \ge 0 \right\} ,$$

whose $(n-2)-$dimensional faces $\mathfrak{F}_1, \ldots, \mathfrak{F}_{n-1}$ are given as

$$\mathfrak{F}_i := \left\{ \sum_{k=1}^{n-1} x_k e_k : x_k \ge 0 \text{ for } k = 1, \ldots, n-1, \, x_i = 0 \right\} ; \quad 1 \le i \le n-1 \, .$$

Let us denote the $(n-3)-$dimensional faces of intersection by $\mathfrak{F}^o_{ij} := \mathfrak{F}_i \cap \mathfrak{F}_j$ for $1 \le i < j \le n-1$ and their union by $\mathfrak{F}^o := \cup_{1 \le i < j \le n-1} \mathfrak{F}^o_{ij}$.

For $n \ge 3$, we shall define the $(n-1)-$dimensional reflected Brownian motion $Y(\cdot) := \{(Y_1(t), \dots Y_{n-1}(t))' \,;\ t \ge 0\}$ on the orthant $\mathbb{R}^{n-1}_+$ with zero drift, constant $((n-1) \times (n-1))$ constant variance/covariance matrix $\mathfrak{A} := \Sigma\Sigma'$, and reflection along the faces of the boundary along constant directions, by

$$
\begin{aligned}
&Y(t) = Y(0) + \Sigma B(t) + \mathfrak{R} L(t)\,; \quad 0 \le t < \infty\,, \\
&\qquad\qquad\qquad\qquad\qquad\qquad\quad Y(0) \in \mathbb{R}^{n-1}_+ \setminus \mathfrak{F}^o\ .
\end{aligned}
\tag{3.13}
$$

Here, $\{B(t);\, 0 \le t < \infty\}$ is $(n-1)-$dimensional standard Brownian motion starting at the origin of $\mathbb{R}^{n-1}$. The $((n-1) \times (n-1))$ reflection matrix $\mathfrak{R}$ has all its diagonal elements equal to one, and spectral radius strictly smaller than one. Finally the components of the $(n-1)-$dimensional process $L(t) := (L_1(t), \dots, L_{n-1}(t));\, 0 \le t < \infty$ are adapted, non-decreasing, continuous and satisfy $\int_0^\infty Y_i(t)\, dL_i(t) = 0$ (that is, $L_i(\cdot)$ is flat off the set $\{t \ge 0 : Y_i(t) = 0\}$) almost surely, for each $i = 1, \dots n-1$. Note that, if $Y(t)$ lies on $\mathfrak{F}^o_{ij} = \mathfrak{F}_i \cap \mathfrak{F}_j$, then $Y_i(t) = Y_j(t) = 0$ for $1 \le i \ne j \le n-1$.

3.2.2 Rotation and Rescaling

Assume that the constant matrix $\mathfrak{A}$ is positive-definite. Let U be the unitary matrix whose columns are the orthonormal eigenvectors of the covariance matrix $\mathfrak{A} = \Sigma\Sigma'$, and let $\mathfrak{L}$ be the corresponding diagonal matrix of eigenvalues such that $\mathfrak{L} = U'\mathfrak{A}\, U$. Note that all the eigenvalues of $\mathfrak{A}$ are positive. Define $\widetilde{Y}(\cdot) := \mathfrak{L}^{-1/2} U Y(\cdot)$. By this rotation and rescaling, we obtain

$$
\widetilde{Y}(t) = \widetilde{Y}(0) + \widetilde{B}(t) + \mathfrak{L}^{-1/2} U \mathfrak{R} L(t)\,; \quad 0 \le t < \infty
\tag{3.14}
$$

from (3.13), where $\widetilde{B}(\cdot) := \mathfrak{L}^{-1/2} U \Sigma B(\cdot)$ is another standard $(n-1)-$dimensional Brownian motion. We may regard $\widetilde{Y}(\cdot)$ as reflected Brownian motion in a new state-space $\widetilde{\mathfrak{S}} := \mathfrak{L}^{-1/2} U \mathbb{R}^{n-1}_+$. The transformed reflection matrix $\widetilde{\mathfrak{R}} := \mathfrak{L}^{-1/2} U \mathfrak{R}$ can be written as

$$
\widetilde{\mathfrak{R}} \,=\, \mathfrak{L}^{-1/2} U \mathfrak{R} \,=\, (\widetilde{\mathfrak{N}} + \widetilde{\mathfrak{Q}})\mathfrak{C} \,=\, (\widetilde{\mathfrak{r}}_1, \dots, \widetilde{\mathfrak{r}}_{n-1})\,,
\tag{3.15}
$$

where

$$
(3.16)\qquad
\begin{aligned}
&\mathfrak{C} := \mathfrak{D}^{-1/2}\,, \quad \mathfrak{D} := diag(\mathfrak{A})\,, \quad \widetilde{\mathfrak{N}} := \mathfrak{L}^{1/2} U \mathfrak{C} \equiv (\widetilde{\mathfrak{n}}_1, \ldots, \widetilde{\mathfrak{n}}_{n-1})\ ,\\
&\widetilde{\mathfrak{Q}} := \mathfrak{L}^{-1/2} U \mathfrak{R} \mathfrak{C}^{-1} - \widetilde{\mathfrak{N}} \equiv (\widetilde{\mathfrak{q}}_1, \ldots, \widetilde{\mathfrak{q}}_{n-1})\,.
\end{aligned}
$$

Here $\mathfrak{D} = diag(\mathfrak{A})$ is the $((n-1)\times(n-1))$ diagonal matrix with the same diagonal elements as those of $\mathfrak{A} = \Sigma\Sigma'$ (the variances). The constant vectors $\widetilde{\mathfrak{r}}_i, \widetilde{\mathfrak{q}}_i, \widetilde{\mathfrak{n}}_i$, $i = 1, \ldots, n-1$ are $((n-1)\times 1)$ column vectors.

Since U is an orthonormal matrix which rotates the state space $\mathfrak{S} = \mathbb{R}_+^{n-1}$, and $\mathfrak{L}^{1/2}$ is a diagonal matrix which changes the scale in the positive direction, the new state-space $\widetilde{\mathfrak{S}}$ is an $(n-1)-$dimensional polyhedron whose i-th face $\widetilde{\mathfrak{F}}_i := \mathfrak{L}^{-1/2} U \mathfrak{F}_i$ has dimension $(n-2)$, for $i = 1, \ldots, n-1$.

Note that $diag(\widetilde{\mathfrak{N}}'\widetilde{\mathfrak{Q}}) = 0$ and $diag(\widetilde{\mathfrak{N}}'\widetilde{\mathfrak{N}}) = I$, that is, $\widetilde{\mathfrak{n}}_i$ and $\widetilde{\mathfrak{q}}_i$ are orthogonal and $\widetilde{\mathfrak{n}}_i$ is a unit vector, i.e., $\widetilde{\mathfrak{n}}_i'\widetilde{\mathfrak{q}}_i = 0$ and $\widetilde{\mathfrak{n}}_i'\widetilde{\mathfrak{n}}_i = 1$ for $i = 1, \ldots, n-1$. Also note that $\widetilde{\mathfrak{n}}_i$ is the inward unit normal to the new i-th face $\widetilde{\mathfrak{F}}_i$ on which the continuous, non-decreasing process $L_i(\cdot)$ actually increases, for $i = 1, \ldots, n-1$. The i-th face $\widetilde{\mathfrak{F}}_i$ can be written as $\{x \in \widetilde{\mathfrak{S}} : \widetilde{\mathfrak{n}}_i' x = \mathfrak{b}_i\}$ for some $\mathfrak{b}_i \in \mathbb{R}$, for $i = 1, \ldots, n-1$.

Moreover, the i-th column $\widetilde{\mathfrak{r}}_i$ of the new reflection matrix $\widetilde{\mathfrak{R}}$ is decomposed into components that are normal and tangential to $\widetilde{\mathfrak{F}}_i$, i.e., $\widetilde{\mathfrak{r}}_i = \mathfrak{C}_{ii}(\widetilde{\mathfrak{n}}_i + \widetilde{\mathfrak{q}}_i)$ for $i = 1, \ldots n-1$, where $\mathfrak{C}_{ii}$ is the (i,i)-element of the diagonal matrix $\mathfrak{C}$. Note that, since the matrix $\mathfrak{L}^{-1/2}U$ of the transformation is invertible, we obtain

$$
(3.17)\qquad \widetilde{Y}(\cdot) \in \widetilde{\mathfrak{F}}_{ij}^o := \widetilde{\mathfrak{F}}_i \cap \widetilde{\mathfrak{F}}_j \iff Y(\cdot) \in \mathfrak{F}_{ij}^o\,; \quad 1 \le i < j \le n-1\,.
$$

Thus, it suffices to work on the transformed process $\widetilde{Y}(\cdot)$ to obtain (3.10) for $Y(\cdot)$ in (3.9).

3.2.3 Attainability

With (3.17) we consider, for $n = 3$ and $n > 3$ separately, the hitting times

$$
(3.18)\qquad
\begin{aligned}
\tau_{ij} &:= \inf\{t > 0 : Y(t) \in \mathfrak{F}_{ij}^o\,\}\\
&= \inf\{t > 0 : \widetilde{Y}(t) \in \widetilde{\mathfrak{F}}_{ij}^o\,\}\,; \quad 1 \le i \ne j \le n-1\,.
\end{aligned}
$$

First we look at the case $n = 3$, i.e., two-dimensional reflected Brownian motion and the hitting time τ_{12}. The directions of reflection $\widetilde{\mathfrak{r}}_1$ and $\widetilde{\mathfrak{r}}_2$ can be written in terms of angles. Note that the angle ξ of the two-dimensional wedge $\widetilde{\mathfrak{S}}$ is positive and smaller than π, since all the eigenvalues of $\mathfrak{A}$ are positive. Let θ_1 and θ_2 with $-\pi/2 < \theta_1,\, \theta_2 < \pi/2$ be the angles between $\widetilde{\mathfrak{n}}_1$ and $\widetilde{\mathfrak{r}}_1$ and between $\widetilde{\mathfrak{n}}_2$ and $\widetilde{\mathfrak{r}}_2$, respectively, measured in such a way that θ_1 is positive if and only if $\widetilde{\mathfrak{r}}_1$ points towards the corner with local coördinate $(0,0)'$ and similar for θ_2.

Paraphrasing Lemma 2.3 in Chapter 2 for Brownian motion reflected on the *two*-dimensional wedge, we obtain the following dichotomous result on the relationship between the stopping time and the sum $\theta_i + \theta_j$ of angles of reflection directions, when $n - 1 = 2$. We shall denote $\widetilde{\mathfrak{F}}^o := \mathfrak{L}^{-1/2} U \mathfrak{F}^o = \bigcup_{1 \le i < j \le n-1} \widetilde{\mathfrak{F}}^o_{ij}$.

Lemma 3.1 ([59])**.** *Suppose that* $\widetilde{Y}(0) = \tilde{y}_0 \in \widetilde{\mathfrak{S}} \setminus \widetilde{\mathfrak{F}}^o$*. If* $\beta := (\theta_1 + \theta_2)/\xi > 0$*, then we have* $\mathbb{P}(\tau_{12} < \infty) = 1$*; if, on the other hand,* $\beta \le 0$*, then we have* $\mathbb{P}(\tau_{12} < \infty) = 0$*.*

In terms of the reflection vectors $\widetilde{\mathfrak{n}}_1, \widetilde{\mathfrak{r}}_1$ and $\widetilde{\mathfrak{n}}_2, \widetilde{\mathfrak{r}}_2$, and with the aid of (3.17) we can cast this result as follows:

Lemma 3.2. *Suppose that* $Y(0) = y_0 \in \mathbb{R}^2 \setminus \mathfrak{F}^o$*. If* $\widetilde{\mathfrak{n}}_1' \widetilde{\mathfrak{q}}_2 + \widetilde{\mathfrak{n}}_2' \widetilde{\mathfrak{q}}_1 > 0$*, then we have* $\mathbb{P}(\tau_{12} < \infty) = 1$*. If, on the other hand,* $\widetilde{\mathfrak{n}}_1' \widetilde{\mathfrak{q}}_2 + \widetilde{\mathfrak{n}}_2' \widetilde{\mathfrak{q}}_1 \le 0$*, then we have* $\mathbb{P}(\tau_{12} < \infty) = 0$*.*

Proof. Now recall the special geometric structure of orthogonality $\widetilde{\mathfrak{n}}_i' \widetilde{\mathfrak{q}}_i = 0$ and $\|\widetilde{\mathfrak{n}}_i\| = 1$ and observe that

$$\left(\widetilde{\mathfrak{N}}' \widetilde{\mathfrak{Q}} + \widetilde{\mathfrak{Q}}' \widetilde{\mathfrak{N}}\right)_{ij} \underset{<}{\overset{\ge}{}} 0 \quad \Longleftrightarrow \quad \widetilde{\mathfrak{n}}_i' \widetilde{\mathfrak{q}}_j + \widetilde{\mathfrak{n}}_j' \widetilde{\mathfrak{q}}_i \underset{<}{\overset{\ge}{}} 0\,; \qquad \forall\ (i,j). \tag{3.19}$$

Note that if $n = 3$, i.e. $n - 1 = 2$, then $\widetilde{\mathfrak{n}}_i' \widetilde{\mathfrak{q}}_j = \|\widehat{\mathfrak{q}}_j\|\, sgn(-\theta_j) \sin(\xi)$ for $1 \le i \ne j \le 2$, where $sgn(x) := 1_{\{x>0\}} - 1_{\{x<0\}}$. The length $\|\widetilde{\mathfrak{q}}_2\|$ of $\widetilde{\mathfrak{q}}_2$ determines the angle θ_2 and vice versa, i.e.,

$$\|\widetilde{\mathfrak{q}}_i\| \underset{<}{\overset{\ge}{}} \|\widetilde{\mathfrak{q}}_j\| \quad \Longleftrightarrow \quad |\theta_i| \underset{<}{\overset{\ge}{}} |\theta_j|.$$

With this and $0 < \xi_{ij} < \pi$, $\sin(\xi_{ij}) > 0$, we obtain

(3.20)
$$\begin{aligned} &\widetilde{\mathfrak{n}}_i'\widetilde{\mathfrak{q}}_j + \widetilde{\mathfrak{n}}_j'\widetilde{\mathfrak{q}}_i = \sin(\xi)(\|\widetilde{\mathfrak{q}}_j\|\, sgn(-\theta_j) + \|\widetilde{\mathfrak{q}}_i\|\, sgn(-\theta_i)) \begin{matrix} \geq \\ = \\ < \end{matrix} 0 \\ &\Longleftrightarrow \beta = (\theta_i + \theta_j)/\xi \begin{matrix} \leq \\ = \\ > \end{matrix} 0; \quad 1 \leq i \neq j \leq 2\,. \end{aligned}$$

Thus, we apply Lemma 3.1 and obtain Lemma 3.2. $\square$

We consider the general case $n > 3$ next. With (3.17) and Proposition 2.4 in Chapter 2 we obtain the following Lemma 3.3, valid for $n \geq 3$.

Lemma 3.3. *Suppose that* $Y(0) = y \in \mathbb{R}_+^{n-1} \setminus \mathfrak{F}^o$ *and* $n \geq 3$, *and that the so-called* skew-symmetry condition

(3.21)
$$\widetilde{\mathfrak{n}}_i'\widetilde{\mathfrak{q}}_j + \widetilde{\mathfrak{n}}_j'\widetilde{\mathfrak{q}}_i = 0\,; \quad 1 \leq i < j \leq n-1$$

holds. Then we have

$$\mathbb{P}_y(\tau < \infty) = 0\,, \qquad \textit{where} \quad \tau := \inf\{t > 0 : Y(t) \in \mathfrak{F}^o\,\}\,.$$

Moreover, the components of adapted non-decreasing continuous process $L(\cdot)$ *defined in* (3.13) *are identified as the local times of one-dimensional processes at level zero:*

$$2L_i(t) = Y_i(t) - Y_i(0) - \int_0^t sgn(Y_i(s))\, dY_i(s)\,; \quad 0 \leq t < \infty\,, \quad i = 1, \ldots, n\,.$$

3.2.4 Coëfficients Structure

Next, we consider the case of linearly growing variance coëfficients defined in (3.12), and recall the tri-diagonal matrices $\mathfrak{A} = \widetilde{\Sigma}\widetilde{\Sigma}'$ as in (3.26) and $\mathfrak{R}$ as in (3.28). Consider the $(n-1)$-dimensional reflected Brownian motion $Y(\cdot)$ defined in (3.13) with $\Sigma = \widetilde{\Sigma}$ and this above $\mathfrak{R}$. We can verify such a pair $(\widetilde{\Sigma}, \mathfrak{R})$ satisfies the following element-wise equations

(3.22)
$$\big(2\mathfrak{D} - \mathfrak{Q}\mathfrak{D} - \mathfrak{D}\mathfrak{Q} - 2\mathfrak{A}\big)_{i\,j} = 0; \quad 1 \leq i, j \leq n-1,$$

where $\mathfrak{D}$ is the diagonal matrix with the same diagonal elements as $\mathfrak{A}$ of (3.16), and $\mathfrak{Q}$ is the $((n-1)\times(n-1))$ matrix whose first-diagonal elements above and below the main diagonal are all 1/2 and other elements are zeros as in (3.26). In fact, it suffices to see the cases $j = i+1, i = 2, \dots, n-1$. The equalities (3.22) are

$$0 - \frac{1}{2}(\widetilde{\sigma}_i^2 + \widetilde{\sigma}_{i+1}^2) - \frac{1}{2}(\widetilde{\sigma}_{i-1}^2 + \widetilde{\sigma}_i^2) + 2\widetilde{\sigma}_i^2 = 0,$$

or equivalently (3.12)

$$\widetilde{\sigma}_i^2 - \widetilde{\sigma}_{i-1}^2 = \widetilde{\sigma}_{i+1}^2 - \widetilde{\sigma}_i^2; \quad 2 \le i \le n-1.$$

Moreover, the equalities (3.22) are equivalent to $(\widetilde{\mathfrak{N}}'\widetilde{\mathfrak{Q}} + \widetilde{\mathfrak{Q}}'\widetilde{\mathfrak{N}})_{ij} = 0$ in (3.19). In fact, from (3.16) with $\mathfrak{D}^{1/2} = \mathfrak{C}^{-1}$ we compute

$$\begin{aligned}\widetilde{\mathfrak{N}}'\widetilde{\mathfrak{Q}} &= \mathfrak{D}^{-1/2}U'\mathfrak{L}^{1/2}\mathfrak{L}^{-1/2}U\mathfrak{R}\mathfrak{D}^{1/2} - \widetilde{\mathfrak{N}}'\widetilde{\mathfrak{N}} \\ &= \mathfrak{D}^{-1/2}(I - \mathfrak{Q})\mathfrak{D}^{1/2} - \mathfrak{D}^{-1/2}\mathfrak{A}\mathfrak{D}^{-1/2} \\ \widetilde{\mathfrak{N}}'\widetilde{\mathfrak{Q}} + \widetilde{\mathfrak{Q}}'\widetilde{\mathfrak{N}} &= 2I - \mathfrak{D}^{-1/2}\mathfrak{Q}\mathfrak{D}^{1/2} - \mathfrak{D}^{1/2}\mathfrak{Q}\mathfrak{D}^{-1/2} - 2\mathfrak{D}^{-1/2}\mathfrak{A}\mathfrak{D}^{-1/2},\end{aligned}$$

and multiply both from the left and the right by the diagonal matrix $\mathfrak{D}^{1/2}$ whose diagonal elements are all positive:

$$\mathfrak{D}^{1/2}(\widetilde{\mathfrak{N}}'\widetilde{\mathfrak{Q}} + \widetilde{\mathfrak{Q}}'\widetilde{\mathfrak{N}})\mathfrak{D}^{1/2} = 2\mathfrak{D} - \mathfrak{Q}\mathfrak{D} - \mathfrak{D}\mathfrak{Q} - 2\mathfrak{A}. \tag{3.23}$$

The equality in the relation (3.22) is equivalent to the *skew-symmetry condition* of Proposition 2.4 introduced and studied by Harrison & Williams in [19], [61] : $\widetilde{\mathfrak{N}}'\widetilde{\mathfrak{Q}} + \widetilde{\mathfrak{Q}}'\widetilde{\mathfrak{N}} = 0$.

Thus, it follows from (3.19), (3.22) and (3.23) that the reflected Brownian motion Z defined in (3.27), under the assumption of Proposition 3.2, satisfies that any two dimensional process (Z_i, Z_j) *never* attains the corner $(0,0)'$ for $1 \le i < j \le n-1$ i.e.

$$\mathbb{P}(Z_i(t) = Z_j(t) = 0, \exists t > 0, \exists (i,j),\, 1 \le i \ne j \le n) = 0. \tag{3.24}$$

Using this fact, we construct a weak solution to (3.38) from the reflected Brownian motion. This final step is explained as an application to the financial Atlas model in the last part of Section 3.2.4.

Now we are ready to show Proposition 3.2. As in Remark 3.2, we can assume that there is no drift, i.e., $b(\cdot) \equiv 0$. Let us start by observing that the dynamics of the sum $\mathfrak{X}(t) := X_1(\cdot) + \cdots + X_n(\cdot)$ can be written as

$$d\,\mathfrak{X}(t) = \sum_{i=1}^{n}\sum_{k=1}^{n} \tilde{\sigma}_k 1_{Q_k^{(i)}}(X(t))d\,W_i(t) = \sum_{k=1}^{n} \tilde{\sigma}_k \, d\,B_k(t); \quad 0 \le t < \infty, \tag{3.25}$$

where $B(\cdot) := \{(B_1(t), \cdots, B_n(t))', 0 \le t < \infty\}$ is an $n-$dimensional Brownian motion starting at the origin, by Dambis-Dubins-Schwarz theorem, with components $B_k(t) := \sum_{i=1}^{n} \int_0^t 1_{Q_k^{(i)}}(X(s))d\,W_i(s)$ for $1 \le k \le n$, $0 \le t < \infty$. In fact, $\langle B_k, B_\ell \rangle = t\,\delta_{k,\ell}$ implies that they are independent standard Brownian motions.

Next, $\widetilde{\Sigma}$ be the $(n-1) \times n$ triangular matrix with entries

$$\widetilde{\Sigma} := \begin{pmatrix} \tilde{\sigma}_1 & -\tilde{\sigma}_2 & & & \\ & \tilde{\sigma}_2 & -\tilde{\sigma}_3 & & \\ & & \ddots & \ddots & \\ & & & \tilde{\sigma}_{n-1} & \tilde{\sigma}_n \end{pmatrix},$$

where the elements in the lower-triangular part and the upper-triangular part, except the first diagonal above the main diagonal, are zeros. Then the process $\{\widetilde{\Sigma}B(t), 0 \le t < \infty\}$ is an $(n-1)-$dimensional Brownian motion starting at the origin of $\mathbb{R}^{n-1}$, with variance/covariance matrix

$$\mathfrak{A} := \widetilde{\Sigma}\widetilde{\Sigma}' := \begin{pmatrix} \tilde{\sigma}_1^2 + \tilde{\sigma}_2^2 & -\tilde{\sigma}_2^2 & & & \\ -\tilde{\sigma}_2^2 & \tilde{\sigma}_2^2 + \tilde{\sigma}_3^2 & \ddots & & \\ & \ddots & \ddots & -\sigma_{n-1}^2 & \\ & & -\tilde{\sigma}_{n-1}^2 & \tilde{\sigma}_{n-1}^2 + \tilde{\sigma}_n^2 & \end{pmatrix}, \tag{3.26}$$

but without drift components.

Now we construct as in Section 3.2.1 an $(n-1)-$dimensional Brownian motion with reflection $Z(\cdot) := \{(Z_1(t), \cdots, Z_{n-1}(t))', 0 \le t < \infty\}$ on $\mathbb{R}_+^{n-1}$ by

$$\begin{aligned} Z_k(t) \; := \; & \tilde{\sigma}_k B_k(t) - \tilde{\sigma}_{k+1} B_{k+1}(t) \\ & + \Lambda^{k,k+1}(t) - \frac{1}{2}\big(\Lambda^{k-1,k}(t) + \Lambda^{k+1,k+2}(t)\big); \quad 0 \le t < \infty \end{aligned} \tag{3.27}$$

for $k = 1, \cdots, n-1$. Here $\Lambda^{k,k+1}(\cdot)$ is a continuous, adapted and non-decreasing process with

$\Lambda^{k,k+1}(0) = 0$ and $\int_0^\infty Z_k(t)\, d\Lambda^{k,k+1}(t) = 0$ almost surely. Setting $\Lambda^{0,1}(\cdot) \equiv \Lambda^{n,n+1}(\cdot) \equiv 0$ for notational convenience, we write in matrix form:

$$Z(t) \;=\; \widetilde{\Sigma}\, B(t) + \mathfrak{R}\, \Lambda(t); \quad 0 \le t < \infty\,,$$

where $\Lambda(\cdot) = (\Lambda^{1,2}(\cdot), \cdots, \Lambda^{k-1,k}(\cdot))'$ and the reflection matrix $\mathfrak{R} = I - \mathfrak{Q}$ is given by

$$\mathfrak{R} \;=\; I - \mathfrak{Q} \;:=\; \begin{pmatrix} 1 & -1/2 & & & \\ -1/2 & 1 & \ddots & & \\ & \ddots & \ddots & \ddots & \\ & & \ddots & \ddots & -1/2 \\ & & & -1/2 & 1 \end{pmatrix}. \tag{3.28}$$

Recall that *if the process $X(\cdot)$ has no "triple collisions"*, then it follows from (3.5) that

$$d\,X_{(k)}(t) = \sum_{i=1}^{n} \mathbf{1}_{\{X_i(t) = X_{(k)}(t)\}} d\,X_i(t) + \frac{1}{2}\Big(d\Lambda^{k,k+1}(t) - d\Lambda^{k-1,k}(t) \Big)\,,$$

for $0 \le t < \infty$. Hence by substituting (3.1) with (3.11) but without drifts into this equation and subtracting, we obtain that

$$X_{(k)}(t) - X_{(k+1)}(t) = Z_k(t); \quad 1 \le k \le n-1,\ 0 \le t < \infty\,, \tag{3.29}$$

and that $\Lambda^{k,k+1}(\cdot)$ is the local time of the one-dimensional process $Z_k(t)$ at level zero for $k = 1, \ldots n-1$. In general, the process $X(\cdot)$ may have triple or more collisions so that we have additional terms in (3.29):

$$X_{(k)}(t) - X_{(k+1)}(t) = Z_k(t) + \zeta_k(t)\,, \quad 1 \le k \le n-1,\ 0 \le t < \infty\,, \tag{3.30}$$

where the contribution $\zeta(\cdot) := (\zeta_1(\cdot), \ldots, \zeta_{n-1}(\cdot))$ from the triple or more collisions can be written for $1 \le k \le n-1$, $0 \le t < \infty$ as

$$\begin{aligned} \zeta_k(t) = & \sum_{\ell=3}^{n} \ell^{-1} \Big[\sum_{j=k+2}^{n} \Lambda^{k,j}(t) - \sum_{j=1}^{k-2} \Lambda^{j,k}(t) \Big] \mathbf{1}_{\{N_k(t) = \ell\}} \\ & - \sum_{\ell=3}^{n} \ell^{-1} \Big[\sum_{j=k+3}^{n} \Lambda^{k+1,j}(t) - \sum_{j=1}^{k-1} \Lambda^{j,k+1}(t) \Big] \mathbf{1}_{\{N_{k+1}(t) = \ell\}}\,. \end{aligned}$$

Remark 3.3. Note that $\zeta(\cdot)$ consists of the (random) linear combination of the local times from collisions of three or more particles, and hence it is *flat*, unless there are triple collisions, i.e., $\int_0^t \mathbf{1}_{\mathfrak{B}^c}(X(s))\, d\zeta(s) = 0$ for $0 \le t < \infty$, where the set $\mathfrak{B}$ is defined as $\{s \ge 0 : X_i(s) = X_j(s) = X_k(s)$ for some $1 \le i < j < k \le n\}$. We use this fact with Lemma 3.4 in the next subsection. □

Application of Lemma 3.3

Under the assumption of Proposition 3.2 we can apply Lemma 3.3 to obtain

$$\mathbb{P}(Z_i(t) = Z_j(t) = 0\,, \exists t > 0\,, \exists (i,j)\,,\ 1 \le i \ne j \le n) = 0\,. \tag{3.31}$$

Thus, $Z(\cdot)$ is a special case of reflected Brownian motion whose each $\Lambda^{k,k+1}(\cdot)$ of non-decreasing finite variation part is exactly the local time of $Z_k(\cdot)$ at level zero.

Now let us state the following lemma to examine the local times from collisions of three or more particles.

Lemma 3.4 ([23])**.** *Let $\alpha(\cdot) = \{\alpha(t)\,;\, 0 \le t < \infty\}$ be a non-negative continuous function with decomposition $\alpha(t) = \beta(t) + \gamma(t)$, where $\beta(\cdot)$ is a strictly positive continuous function and $\gamma(\cdot)$ is a continuous function that can be written as a difference of two non-decreasing functions which are flat off $\{t \ge 0 : \alpha(t) = 0\}$, i.e., $\int_0^t \mathbf{1}_{\{\alpha(s)>0\}} d\gamma(s) = 0$ for $0 \le t < \infty$. Assume that $\gamma(0) = 0$ and $\alpha(0) = \beta(0) > 0$. Then, $\gamma(t) = 0$ and $\alpha(t) = \beta(t)$ for $0 \le t < \infty$.*

Proof. Let us fix arbitrary $T \in [0, \infty)$. Since $\beta(\cdot)$ is strictly positive, we cannot have simultaneously $\alpha(t) = \beta(t) + \gamma(t) = 0$, and $\gamma(t) \ge 0$. Because the continuous function $\beta(\cdot)$ attains the minimum on $[0, T]$, we obtain

$$\begin{aligned} \{t \in [0,T] : \alpha(t) = 0\} &= \{t \in [0,T] : \alpha(t) = 0, \gamma(t) < 0\} \\ &\subset \{t \in [0,T] : \gamma(t) \le -\min_{0 \le s \le T} \beta(s) < 0\}. \end{aligned} \tag{3.32}$$

Let us define $t_0 := \inf\{t \in [0,T] : \alpha(t) = 0\}$ following the usual convention that if the set is empty, $t_0 := \infty$. If $t_0 = \infty$, then $\alpha(t) > 0$ for $0 \le t < \infty$ and hence it follows from the assumptions $\gamma(0) = 0$ and $\int_0^T \mathbf{1}_{\{\alpha(t)>0\}} d\gamma(t) = 0$ for $0 \le T < \infty$ that $\gamma(\cdot) \equiv 0$. On the other hand, if $t_0 < \infty$, then it follows from the same argument as in (3.32) that $\gamma(t_0) < -\min_{0 \le s \le t_0} \beta(s) < 0$. However, this is impossible, since $\alpha(s) > 0$ for $0 \le s < t_0$ by the definition of t_0 and hence the continuous function $\gamma(\cdot)$ is flat on $[0, t_0)$, i.e., $0 = \gamma(0) = \gamma(t_0-) = \gamma(t_0)$. Thus, $t_0 = \infty$ and $\gamma(\cdot) \equiv 0$. Therefore, the conclusions of Lemma 3.4 hold.

$\square$

Under the assumption of Proposition 3.2, applying the above Lemma 3.4 with (3.30), (3.31) and $\alpha(\cdot) = X_{(k)}(\cdot, \omega) - X_{(k+2)}(\cdot, \omega)$, $\beta(\cdot) = Z_k(\cdot, \omega) + Z_{k+1}(\cdot, \omega)$ and $\gamma(\cdot) = \zeta_k(\cdot, \omega) + \zeta_{k+1}(\cdot, \omega)$ for $\omega \in \Omega$, we obtain $\alpha(\cdot) = \beta(\cdot)$, i.e.,

$$X_{(k)}(\cdot) - X_{(k+2)}(\cdot) = Z_k(\cdot) + Z_{k+1}(\cdot), \quad k = 1, \ldots, n-2. \tag{3.33}$$

See Remark 3.3. Combining (3.33) with (3.31), we obtain $X_{(k)}(\cdot) - X_{(k+2)}(\cdot) > 0$ or

$$\mathbb{P}(X_{(k)}(t) = X_{(k+1)}(t) = X_{(k+2)}(t), \exists t > 0, \exists k, 1 \le k \le n-2) = 0. \tag{3.34}$$

Therefore, there are "*no triple collisions*" under the assumption of Proposition 3.2. This concludes the proof of Proposition 3.2.

In summary, we *recover* the n−dimensional ranked process $X_{(\cdot)}$ of X by considering the linear transformation. Specifically, construct n-dimensional ranked process

$$\Psi_{(\cdot)}(t) := (\Psi_{(1)}(t), \cdots, \Psi_{(n)}(t)); \quad 0 \le t < \infty$$

from the sum $\mathfrak{X}(t)$, $0 \le t < \infty$ defined in (3.25) and the reflected Brownian motion $Z(\cdot)$, such that the differences satisfy

$$\Psi_{(k)}(t) - \Psi_{(k+1)}(t) = Z_k(t), \quad k = 1, \ldots n-1, \tag{3.35}$$

and the sum satisfies

$$\sum_{k=1}^{n} \Psi_{(k)}(t) = \mathfrak{X}(t); \quad 0 \le t < \infty. \tag{3.36}$$

Each element is uniquely determined by

$$\begin{pmatrix} \Psi_{(1)}(\cdot) \\ \Psi_{(2)}(\cdot) \\ \vdots \\ \Psi_{(n)}(\cdot) \end{pmatrix} = \frac{1}{n} \begin{pmatrix} \mathfrak{X}(\cdot) + Z_{n-1}(\cdot) + (n-2)Z_{n-2}(\cdot) + \cdots + (n-1)Z_1(\cdot) \\ \mathfrak{X}(\cdot) + Z_{n-1}(\cdot) + (n-2)Z_{n-2}(\cdot) + \cdots - Z_1(\cdot) \\ \vdots \\ \mathfrak{X}(\cdot) - (n-1)Z_{n-1}(\cdot) - (n-2)Z_{n-2}(\cdot) - \cdots - Z_1(\cdot) \end{pmatrix}$$

for $0 \le t < \infty$. Under the assumption of Proposition 3.2, we obtain (3.31) and hence with (3.35) we arrive at

$$\mathbb{P}(\Psi_{(k)}(t) = \Psi_{(k+1)}(t) = \Psi_{(k+2)}(t),\ \exists t > 0,\ 1 \le \exists k \le n-2) = 0\,, \tag{3.37}$$

in the same way as discussed in (3.10).

Thus, the ranked process $\{X_{(\cdot)}(t), 0 \le t < \infty\}$ of the original process $X(\cdot)$ without collision of three or more particles, and the ranked process $\Psi_{(\cdot)}(\cdot)$ defined in the above, are equivalent, since both of them have the same sum (3.36) and the same non-negative difference processes $Z(\cdot)$ identified in (3.29) and (3.35). Then, we may view $\Psi_{(\cdot)}(\cdot)$ as the weak solution to the SDE for the ranked process $X_{(\cdot)}(\cdot)$. Finally, we define $\Psi(\cdot) := (\Psi_1(\cdot), \ldots, \Psi_n(\cdot))$ where $\Psi_i(\cdot) = \Psi_{(p_t^{-1}(i))}(\cdot)$ for $i = 1, \ldots, n$, and $p_t^{-1}(i)$ is defined in (3.7). Then, $\Psi(\cdot)$ is the weak solution of SDE (3.38). This construction of solution leads us to the invariant properties of the Atlas model given in [5] and [22].

In the next section we shall discuss some details of the resulting model, as an application of Proposition 3.2.

3.3 Application to Hybrid Atlas Model for an Equity Market

Let us introduce the *hybrid* Atlas model

$$\begin{aligned} d\,X_i(t) &= \left(\sum_{k=1}^{n} g_k 1_{Q_k^{(i)}}(X(t)) + \gamma_i + \gamma \right) d\,t + \sum_{k=1}^{n} \tilde{\sigma}_k 1_{Q_k^{(i)}}(X(t)) d\,W_i(t); \\ &\text{for } 1 \le i \le n,\ 0 \le t < \infty, \qquad (X_1(0), \ldots, X_n(0))' = x_0 \in \mathbb{R}^n\,. \end{aligned} \tag{3.38}$$

A special case $\gamma_i \equiv 0$ was introduced by Fernholz [12] and studied by Banner, Fernholz & Karatzas [5]. Here $X(\cdot) = (X_1(\cdot), \cdots, X_n(\cdot))'$ represents the vector of logarithmic function of asset capitalizations in an equity market. We assume that $\tilde{\sigma}_k > 0$ and $g_k\,,\ \gamma_i\ \ i\,, k = 1, \cdots, n$ are constants satisfying the conditions

$$\sum_{i=1}^{n} \gamma_i + \sum_{k=1}^{n} g_k = 0\,. \tag{3.39}$$

imposed to ensure that the resulting diffusion $X(\cdot)$ has some ergodic properties.

3.3.1 Brownian Motion with Reflection

Define $(n-1)-$dimensional process $Y = ((Y_1(t), \ldots, Y_{n-1}(t))\,, 0 \le t < \infty)$ by $Y(t) := \mathcal{P}X(t)$ where $\mathcal{P}$ is the projection operator defined in (1.3) for $0 \le t < \infty$.

Fernholz [12] showed the following Lemma.

Proposition 3.3 (Fernholz [12]). *If the process $X(\cdot)$ satisfies path-wise mutually degenerate conditions:*

$$\begin{aligned} &\mathbb{P}_{x_0}\left(Leb\{0 \le t \le T : X_i(t) = X_j(t)\} = 0\right) = 1 \qquad \textit{and} \\ &\mathbb{P}_{x_0}(X_i(t) = X_j(t) = X_k(t) \quad \textit{i.o.}) = 0\,, \end{aligned} \tag{3.40}$$

then the $(n-1)-$dimensional projected process $Y(\cdot) := \mathcal{P}\,X(\cdot)$ satisfies the following stochastic differential equation:

$$d\,Y(t) = d\,(\mathcal{P}X(t)) = \mu(X(t))\,d\,t + \sigma(X(t))\,d\,W(t) + (I - \mathfrak{P})\,d\,L(t)\,; \tag{3.41}$$

for $0 \le t < \infty$, where $\mu(\cdot)$ and $\sigma(\cdot)$ are again piece-wise constant functions given by

$$\begin{aligned} \mu(x) &= \sum_{\pi \in \Pi} b_\pi \sum_{i=1}^{n-1} (\mathbf{1}_{\{p^x(k)=i\}} - \mathbf{1}_{\{p^x(k+1)=i\}}) 1_{\mathcal{R}_\pi}(x), \\ \sigma(x) &= \sum_{\pi \in \Pi} \sigma_\pi \sum_{i=1}^{n-1} (\mathbf{1}_{\{p^x(k)=i\}} - \mathbf{1}_{\{p^x(k+1)=i\}}) 1_{\mathcal{R}_\pi}(x)\,; \quad x \in \mathbb{R}^n\,. \end{aligned} \tag{3.42}$$

In the third term $L = ((L_1(t), \ldots, L_{n-1}(t))', t \in \mathbb{R}_+)$ of (3.41), $L_j(t)$ is the local time of $Y_j(t)$ at 0 for $j = 1, \ldots, n-1$, $0 \le t < \infty$. $\mathfrak{R} := I - \mathfrak{P}$ is $((n-1)\times(n-1))$ matrix reflection matrix where the elements of $\mathfrak{P}$ are nonnegative and its diagonal elements are zero, and its spectral is *less than unity, and* I *is the* $(n-1)-$*dimensional identity matrix.*

In Section 3.2 we see that the linearly growing variance condition (3.12) is sufficient for (3.40). The first condition is satisfied from the property of non-degenerate Brownian motion. Moreover, using the results of Brownian motion with reflection in Theorem 2.6, the invariant distribution of hybrid Atlas model under the linearly growing variance condition has a product form density.

Basic Adjoint Relations

Under the linearly growing condition (3.12) the projection $Y(\cdot) = \mathcal{P}X(\cdot)$ of $n-$dimensional process $X(\cdot)$ can be identified as the reflected Brownian motion on $\mathfrak{S} := (\mathbb{R}_+)^{n-1}$ defined by $dY(t) = \mu(X(t))\,dt + \Sigma\,dW(t) + \mathfrak{R}\,dL(t)$ for $0 \le t < \infty$. The diffusion part $\mu(X(t))dt + \Sigma dW(t)$ has the piece-wise constant coefficients. Define the infinitesimal generator $\bar{\mathcal{A}}$ on $C_b^2(\mathfrak{S})$ by

$$\bar{\mathcal{A}}\,\varphi(y) := \bar{\mathcal{A}}\varphi(\mathcal{P}x) := \frac{1}{2}\sum_{i,j=1}^{n-1} \bar{a}_{ij}(x)\frac{\partial^2\varphi(y)}{\partial y_i \partial y_j} + \sum_{i=1}^{n-1}\mu_i(x)\frac{\partial\varphi(y)}{\partial y_i}\,; \quad \varphi \in C_b^2(\mathbb{R}_+^{n-1})\,, \tag{3.43}$$

where $\mathfrak{A} := (\bar{a}_{ij})_{1\le i,j,\le n-1} = \Sigma\Sigma'$.

By an application of Ito's formula and the properties of additive functionals we obtain the following Lemma 3.5 with the same reasoning in Section 2.4.5.

Lemma 3.5. *Suppose that the stationary distribution* $\nu^*(\cdot)$ *of process* $Y(\cdot)$ *on* $\mathfrak{S}$ *exists. For* $f \in C_b^2(\mathfrak{S})$ *we have the so called basic adjoint relation* (BAR)*:*

$$\int_{x\in\mathbb{R}^n} \bar{\mathcal{A}}\,f(\mathcal{P}x)\,d\nu^*(\mathcal{P}\,x) + \frac{1}{2}\sum_{k=1}^{n-1}\int_{\mathcal{P}\,x\in\mathfrak{F}_k} \frac{\partial f(y)}{\partial y_k}\Big|_{y=\mathcal{P}\,x}\,d\,v(\,\mathcal{P}\,x) = 0\,, \tag{3.44}$$

where $\bar{\mathcal{A}}\,f(\mathcal{P}\,x)$ *is defined by* (3.43)*.*

Stationary Distribution of Difference of Rankings

Define a class $\mathcal{C}^\mu$ of piece-wise constant function by

$$\{c(x) : \mathbb{R}^n \to \mathbb{R}^{n-1} \mid c(x) = J\mu(x), \text{ for some invertible } ((n-1)\times(n-1)) \text{ matrix } J\,\} \tag{3.45}$$

The following Lemma 3.6 is obtained through the same argument as Proposition 2.7.

Lemma 3.6. *The linearly growing variance condition* (3.12) *holds if and only if there exists a non-zero function* $\wp(\mathcal{P}x) = C\exp[-\langle c(x)\,,\,\mathcal{P}x\,\rangle\,]$*, satisfies the BAR* (3.44) *with* $d\,\nu^*(\mathcal{P}x) =$

$\wp(\mathcal{P}x)\,dx$, where $c(\cdot) \in \mathcal{C}^{\mu}$ in (3.45) and C is a positive normalizing constant. Moreover, $c(\cdot) \in \mathcal{C}^{\mu}$ is uniquely determined for each $\mu(\cdot)$ by

$$c(x) = 2\, diag\,(\mathfrak{R})[\, diag\,(\mathfrak{A})\,]^{-1}\mathfrak{R}^{-1}\mu(\,\mathcal{P}\,x\,)\,; \quad x \in \mathbb{R}^n\,, \tag{3.46}$$

where diag $(\mathfrak{R})$ and diag $(\mathfrak{A})$ are the $((n-1)\times(n-1))$ diagonal matrix whose diagonal elements are the same as $\mathfrak{R}$ and $\mathfrak{A}$, respectively.

Combining there Lemmata 3.5 and 3.6 with Proposition 2.8 together, we obtain the following Theorem 3.1 which is a parallel statement of Theorem 2.6. The stability condition $\mathfrak{R}^{-1}\,b(\cdot) < 0$ elementwise in $\mathfrak{S}$ of Theorem 2.6 can be written as $\mathfrak{R}^{-1}\,\mu(\cdot) < 0$ in $\mathbb{R}^n$ now. The following Lemma 3.7 states this condition in terms of the drift coëfficients $\gamma\,,\gamma_i\,,g_k\,,\ 1 \le i\,,k \le n\,.$

Lemma 3.7. *In the hybrid Atlas model* (3.38) *with* (3.39) *assume that the linearly growing condition* (3.12) *and the following stability condition hold:*

$$\sum_{\ell=1}^{k}(g_\ell + \gamma_{\pi(\ell)}) < 0\,, \quad k = 1, \ldots, n-1,\ \pi \in \Pi\,, \tag{3.47}$$

then $\mathfrak{R}^{-1}\mu(x) < 0$ for $x \in \mathbb{R}^n$ and the process $Y(\cdot)$ has the stationary distribution.

Theorem 3.1. *Suppose that the Hybrid Atlas model* (3.38) *satisfies the linearly growing variance condition* (3.12)*, and the stability conditions* (3.39) *and* (3.47)*. Then, $Y(\cdot) = \mathcal{P}X(\cdot)$ has the product form stationary distribution with the density $\wp(\mathcal{P}x)$*

$$\begin{aligned}
&\wp(\mathcal{P}x) := \Big[\sum_{\pi\in\Pi}\prod_{j=1}^{n-1}\big(\widetilde{\nu}_j(\pi)\big)^{-1}\Big]^{-1}\exp\Big(-\sum_{j=1}^{n-1}\widehat{\nu}_j(x)\big(x_{p^x(j)} - x_{p^x(j+1)}\big)\Big)\,,\\
&\widehat{\nu}_j(x) := -\frac{4\sum_{\ell=1}^{j}(g_\ell + \gamma_{p^x(\ell)})}{\widetilde{\sigma}_j^2 + \widetilde{\sigma}_{j+1}^2}\,, \quad \widetilde{\nu}_j(\pi) := -\frac{4\sum_{\ell=1}^{j}(g_\ell + \gamma_{\pi(\ell)})}{\widetilde{\sigma}_j^2 + \widetilde{\sigma}_{j+1}^2}\,;
\end{aligned} \tag{3.48}$$

for $1 \le j \le n-1\,,\ x \in \mathbb{R}^n\,.$

Remark 3.4. The difference process $Y(\cdot) = \mathcal{P}X(\cdot) = (X_{(1)}(\cdot) - X_{(2)}(\cdot), \ldots, X_{(n-1)}(\cdot) - X_{(n)}(\cdot))$ has the invariant stationary distribution, while the process $X(\cdot)$ itself is not stationary. The system represented by $X(\cdot)$ can be seen as n tiny particles marked their locations by $(X_1(\cdot), \ldots, X_n(\cdot))$ diffuse in one-dimensional real line $\mathbb{R}$. Under the assumptions of Theorem 3.1 the particles can move in the whole real line but always stay together, being attracted by their location average $n^{-1}\sum_{i=1}^{n} X_i(\cdot)$. Thus, the range $X_{(1)}(T) - X_{(n)}(T)$ of process can not diverge, as $T \to \infty$, which is a remarkably different behavior from the standard Brownian motions or the market models represented by the Black-Scholes type.

Remark 3.5. When the model (3.38) satisfies $\gamma_i = 0$ for $i = 1, \ldots, n$, the above formula (3.48) is reduced much simpler to

$$\begin{aligned} \wp(\mathcal{P}x) &:= \Big[\prod_{j=1}^{n-1} \bar{\nu}_j\Big] \exp\Big(- \sum_{j=1}^{n-1} \bar{\nu}_j \big(x_{p^x(j)} - x_{p^x(j+1)}\big)\Big)\,, \\ \bar{\nu}_j &:= -\frac{4\sum_{\ell=1}^{j} g_\ell}{\widetilde{\sigma}_j^2 + \widetilde{\sigma}_{j+1}^2}\,; \quad x \in \mathbb{R}^n\,,\ j = 1, \ldots, n-1\,. \end{aligned} \tag{3.49}$$

This is the product of exponential distributions, conjectured by Banner, Fernholz & Karatzas [5]. When $\widetilde{\sigma}_1^2 = \cdots = \widetilde{\sigma}_n^2$ in (3.38), the diffusion coëfficient does not depend on rankings of the process and becomes the standard Brownian motion. Pal & Pitman [46] studied this standard Brownian case of the Atlas model and computed the stationary distribution explicityly. Our result here is consistent with their results. □

Remark 3.6. The stability condition conditions (3.39) and $\mathfrak{R}^{-1}\mu(\cdot) < 0$ make the coëfficients $\widehat{\nu}_j(\cdot)$ and $\bar{\nu}_j(\cdot)$ of exponential distributions in (3.48) and (3.49) positive. In fact, for simplicity, assume $\gamma_i = 0$, $i = 1, \ldots, n$. Then, $\mu_k(\cdot) = g_k - g_{k+1}$ for $k = 1, \ldots, n-1$, and (3.39) implies $\sum_{i=1}^{n} g_i = 0$. Let us write $(n-1)-$dimensional vector $\mu(\cdot) \equiv \underset{\sim}{h} := (g_1 - g_2, \ldots, g_{n-1} - g_n)'$ and $n-$dimensional vector $\underset{\sim}{g} := (g_1, \ldots, g_n)'$. By the following linear algebra:

$$\mathfrak{R}^{-1}\underset{\sim}{h} = \mathfrak{R}^{-1} \begin{pmatrix} 1 & -1 & & & \\ & 1 & -1 & & \\ & & \ddots & -1 & \\ & & & 1 & -1 \end{pmatrix} \begin{pmatrix} g_1 \\ \vdots \\ \vdots \\ g_n \end{pmatrix} \equiv \mathfrak{R}^{-1}\Delta \underset{\sim}{g}$$

where Δ is the $((n-1)\times n)$ matrix and $\underset{\sim}{g}$ is the $(n\times 1)$ vector. Moreover, we readily see that

$$2\mathfrak{R}\begin{pmatrix} 1 & & & \\ \vdots & 1 & & \\ \vdots & & \ddots & \\ 1 & \cdots & \cdots & 1 \end{pmatrix}\begin{pmatrix} g_1 \\ \vdots \\ \vdots \\ g_{n-1} \end{pmatrix} = 2\mathfrak{R}\begin{pmatrix} 1 & & & & 0 \\ \vdots & 1 & & & \vdots \\ \vdots & & \ddots & & \vdots \\ 1 & \cdots & \cdots & 1 & 0 \end{pmatrix}\begin{pmatrix} g_1 \\ \vdots \\ \vdots \\ g_n \end{pmatrix}$$

$$= \begin{pmatrix} 2 & -1 & & & \\ -1 & 2 & \ddots & & \\ & \ddots & \ddots & \ddots & \\ & & \ddots & 2 & -1 \\ & & & -1 & 2 \end{pmatrix}\begin{pmatrix} 1 & & & & 0 \\ \vdots & 1 & & & \vdots \\ \vdots & & \ddots & & \vdots \\ 1 & \cdots & \cdots & 1 & 0 \end{pmatrix}\underset{\sim}{g}$$

$$= \begin{pmatrix} 1 & -1 & & & & & 0 \\ 0 & 1 & \ddots & & & & 0 \\ \vdots & \ddots & \ddots & \ddots & & & \vdots \\ & & \ddots & 1 & -1 & & \\ 0 & \cdots & & 0 & 1 & -1 & 0 \\ 1 & 1 & \cdots & \cdots & 1 & 2 & 0 \end{pmatrix}\underset{\sim}{g} = \begin{pmatrix} 1 & -1 & & & & & 0 \\ 0 & 1 & \ddots & & & & 0 \\ & & \ddots & \ddots & & & \vdots \\ & & & 1 & -1 & & \\ & & & & 1 & -1 & 0 \\ 0 & 0 & \cdots & \cdots & 0 & 1 & -1 \end{pmatrix}\underset{\sim}{g}$$

$$= \Delta\underset{\sim}{g} \quad (\text{ we have used } g_1 + \ldots g_n = 0 \text{ in the fourth equality})$$

imply

$$0 < -\mathfrak{R}^{-1}\underset{\sim}{h} = -\mathfrak{R}^{-1}\Delta\underset{\sim}{g} = -2\begin{pmatrix} 1 & & & \\ \vdots & 1 & & \\ \vdots & & \ddots & \\ 1 & \cdots & \cdots & 1 \end{pmatrix}\begin{pmatrix} g_1 \\ \vdots \\ \vdots \\ g_{n-1} \end{pmatrix}.$$

Thus, the stability condition conditions (3.39) and $\mathfrak{R}^{-1}\mu(\cdot) < 0$ imply the positivity of the coëfficients $\widehat{\nu}_j(\cdot)$ and $\bar{\nu}_j(\cdot)$ in (3.48) and (3.49). The above Lemma 3.7 is shown in the same manner. $\square$

3.3.2 Ergodic Properties

Strong Law of Large Numbers [26].

Because of (3.39), adding up over $i = 1, \dots, n$ in (3.38), we obtain

$$d\Big(\sum_{i=1}^{n} X_i(t)\Big) = n\,\gamma\,d\,t + \sum_{k=1}^{n} \widetilde{\sigma}_k\,d\,B_k(t)\,; \quad 0 \le t < \infty\,, \tag{3.50}$$

where $B(\cdot)$ is defined similarly in (3.25). Note that there is no drift by assumption in (3.25). The Strong law of Large Numbers for Brownian motion gives then

$$\lim_{T\to\infty} \frac{1}{T} \sum_{i=1}^{n} X_i(T) = n\,\gamma\,, \quad a.s. \tag{3.51}$$

Under the same assumptions in Theorem 3.1, this property can be strengthened to

$$\lim_{T\to\infty} \frac{X_i(T)}{T} = \gamma\,, \quad a.s.\,; \quad i = 1, \dots, n\,, \tag{3.52}$$

as well as to

$$\lim_{T\to\infty} \frac{X_{(k)}(T)}{T} = \gamma\,, \quad a.s.\,; \quad k = 1, \dots, n\,. \tag{3.53}$$

But then the elementary inequality $e^{X_{(1)}(T)} \le \sum_{i=1}^{n} e^{X_i(T)} \le n\,e^{X_{(1)}(T)}$ implies

$$\lim_{T\to\infty} \frac{1}{T} \log\Big(\sum_{i=1}^{n} \exp\big(X_i(T)\big)\Big) = \gamma\,, \quad a.s. \tag{3.54}$$

In conjunction with (3.52), this implies that the coherence of the model:

$$\lim_{T\to\infty} \frac{1}{T} \log \frac{\exp(X_i(T))}{\exp(X_1(T)) + \cdots + \exp(X_n(T))} = 0\,, a.s.\,; \quad i = 1, \dots, n\,. \tag{3.55}$$

In the previous Section 3.3 we see the ergodic property of drifted Brownian motion $Y(\cdot)$ with reflection under the conditions (3.39) and (3.47) in Lemma 3.7, and moreover, obtain explicitly the invariant density function (3.48) of $\mathcal{P}X(\cdot)$ in Theorem 3.1 under the additional linearly growing condition (3.12). The ergodic property of $(n-1)-$dimensional process $Y(\cdot)$ leads that of $n-$dimensional process $X(\cdot)$ around its mean $\bar{X} := \{\bar{X}(t), 0 \le t < \infty\}$, where $\bar{X}(\cdot) = n^{-1}\sum_{i=1}^{n} X_i(\cdot)$. For example, if Y is ergodic, the de-meaned process $(X_1(t) - \bar{X}(t), \dots, X_n -$

$\bar{X}(t), t \in \mathbb{R}_+)$ is ergodic. Moreover, if Y is ergodic, so is

$$\left(\frac{X_1(t)}{X_1(t) + \cdots + X_n(t)}, \ldots, \frac{X_n(t)}{X_1(t) + \cdots + X_n(t)}, \, 0 \le t < \infty \right). \tag{3.56}$$

This is one approach toward the ergodic properties through the Brownian motion with reflection. There is another approach to the ergodicity, based on the elegant criteria of Khas'minskiĭ[30], [31]. Here let us see briefly the argument given by Banner, Fernholz, & Karatzas [5] and Karatzas [26].

Recall that Π is the symmetric group of permutations of , $\{1, \ldots n\}$. Let us use the notation of $(n \times 1)$ vectors $\underset{\sim}{g} = (g_1, \ldots, g_n)'$ and $\mathbf{1_n} := (1, \ldots, 1)'$, and functions $G : \mathbb{R}^n \mapsto \mathbb{R}^n$ and $S : \mathbb{R}^n \mapsto \mathbb{R}^n$ defined by

$$G(y) := \sum_{\pi \in \Pi} \mathbf{1}_{\mathcal{R}_\pi}(y) \begin{pmatrix} g_{\pi^{-1}(1)} \\ \vdots \\ g_{\pi^{-1}(n)} \end{pmatrix}, \quad S(y) := \sum_{\pi \in \Pi} \mathbf{1}_{\mathbb{R}_\pi}(y) \text{ diag } (\widetilde{\sigma}_{\pi^{-1}(1)}, \ldots, \widetilde{\sigma}_{\pi^{-1}(n)}) \, . \tag{3.57}$$

We can then write the model (3.38) in the vector format as

$$d\,X(t) = \big(G(Y(t)) + \underset{\sim}{g} + \gamma \mathbf{1_n}\big)\, d\,t + S(Y(t))\, d\,W(t)\,; \quad 0 \le t < \infty\,. \tag{3.58}$$

Note also that, because of (3.39), we have

$$d\,\Big(\sum_{i=1}^{n} X_i(t)\Big) = n\gamma\, d\,t + \mathbf{1_n}' S(Y(t))\, d\,W(t)\,, \tag{3.59}$$

so that the centered process $\widetilde{X}(t) := X(t) - \mathbf{1_n}(\sum_{i=1}^n X_i(t))/n$ satisfies

$$d\,\widetilde{X}(t) = \widetilde{G}(X(t))\, d\,t + \widetilde{S}(X(t))\, d\,W(t)\,, \tag{3.60}$$

where we are setting $\widetilde{G}(y) := G(y) + \underset{\sim}{g}$, $\widetilde{S}(y) := S(y) - n^{-1}\mathbf{1_n}\mathbf{1_n}' S(y)$. Note that the ranks of coördinates are preserved under the shift of all of them by the same scaler amount, i.e., $y \in \mathcal{R}_\pi \Leftrightarrow y + \alpha \mathbf{1_n} \in \mathcal{R}_\pi$, $G(y + \alpha \mathbf{1_n}) = G(y)$, $S(y + \alpha\mathbf{1_n}) = S(y)$ for $y \in \mathbb{R}^n$, $\alpha \in \mathbb{R}$. Then, (3.60) can be written as

$$d\,\widetilde{X}(t) = \widetilde{G}(\widetilde{X}(t))\, d\,t + \widetilde{S}(\widetilde{X}(t))\, d\,W(t)\,; \quad 0 \le t < \infty\,. \tag{3.61}$$

The state space of this centered process $\widetilde{X}(\cdot)$ is the subspace $\mathcal{N} := \{y \in \mathbb{R}^n \,:\, y_1 + \cdots y_n = 0\}$ of $\mathbb{R}^n$, and the vector $\mathbf{1_n}$ of ones is orthogonal to this space.

The crucial observation is that the stability condition of diffusion $\widetilde{X}(\cdot)$:

$$y'(\underset{\sim}{g} + G(y)) \le c\|y\|\,, \quad y \in \mathcal{N} \tag{3.62}$$

holds for a suitable $c < 0$. Indeed, for every $y \in \mathcal{N}$,

$$\begin{aligned} y'(\underset{\sim}{g} + G(y)) &= \sum_{\ell=1}^{n} y_{\pi(\ell)}\gamma_{\pi(\ell)} + \sum_{\ell=1}^{n} y_{\pi(\ell)} g_\ell \\ &= y_{\pi(n)} \sum_{\ell=1}^{n} (g_\ell + \gamma_{\pi(\ell)}) + \sum_{k=1}^{n-1} (y_{\pi(k)} - y_{\pi(k-1)}) \cdot \sum_{\ell=1}^{k} (g_\ell + \gamma_{\pi(\ell)}) \\ &= \sum_{k=1}^{n-1} (y_{\pi(k)} - y_{\pi(k-1)}) \cdot \sum_{\ell=1}^{k} (g_\ell + \gamma_{\pi(\ell)}) \end{aligned} \tag{3.63}$$

holds for some $\pi \in \Pi$ such that $y \in \mathcal{R}_\pi$. Note that $y \in \mathcal{N}$, and $y_{\pi(1)} \ge y_{\pi(2)} \ge \cdots \ge y_{\pi(n)}$ imply $y_{\pi(1)} \ge 0 \ge y_{\pi(n)}$, $y_{\pi(1)} \ge y_i$ for all $1 \le i \le n$. In particular, we have $\|y\|^2 \le n \max(y^2_{\pi(1)}, y^n_{\pi(n)}) \le n(y_{\pi(1)} - y_{\pi(n)})^2$, and thus

$$\begin{aligned} y'(\underset{\sim}{g} + G(y)) &= \sum_{k=1}^{n-1} (y_{\pi(k)} - y_{\pi(k+1)}) \cdot \sum_{\ell=1}^{k} (g_\ell + \gamma_{\pi(\ell)}) \\ &\le c\sqrt{n} \sum_{k=1}^{n-1} (y_{\pi(k)} - y_{\pi(k+1)}) = c\sqrt{n}(y_{\pi(1)} - y_{\pi(n)}) \\ &\le c\|y\|\,; \quad y \in \mathcal{N}\,, \end{aligned} \tag{3.64}$$

where we have set the constant c:

$$c := \frac{1}{\sqrt{n}} \max_{\substack{\le k \le n-1, \\ \pi \in \Pi}} \Big(\sum_{\ell=1}^{k} (g_\ell + \gamma_k \Big) < 0$$

by virtue of condition (3.47). Under the condition (3.62) which we just verifeid, Khas'miniskiĭ's theory tells us that the process $\widetilde{X}(\cdot)$ is recurrent with respect to $B_\delta(0) \cap \mathcal{N}$ for some ball $B_\delta(0)$ of radius $\delta > 0$ centered at the origin. Consequently, the centered process $\widetilde{X}(\cdot)$ has an invariant distribution $\nu(\cdot)$ on $\mathcal{N}$ that satisfies the Strong Law of Large Numbers

$$\lim_{T\to\infty} \frac{1}{T} \int_0^T f(\widetilde{X}(t))\, dt = \int_{\mathcal{N}} f(y)\, \nu(dy) \tag{3.65}$$

for every bounded, measurable $f : \mathcal{N} \mapsto \mathbb{R}$.

Average Occupation Time

For the application to portfolio analysis of equity market we examine the average occupation time of the process $X(\cdot)$ in region $Q_i^{(k)}$, i.e., the average time of $X_i(\cdot)$ being kth ranked : for each $k, i = 1, \ldots, n$. By the Strong Law of Large Numbers (3.65) we obtain

$$\lim_{T\to\infty} \frac{1}{T} \int_0^T 1_{Q_i^{(k)}}(X(t))dt = \nu(\mathcal{N} \cap Q_{(i)}^{(k)}) =: \theta_{ki}, \quad \mathbb{P} - a.s., \tag{3.66}$$

namely, that the long-term averages of occupation times, spent by any given particle $X_i(\cdot)$, $i = 1, \ldots, n$ in any particular rank $k = 1, \ldots, n$, exist and are real numbers, i.e., non-random. Since $\cup_{i=1}^n Q_i^{(k)} = \mathbb{R}^n = \cup_{k=1}^n Q_i^{(k)}$, the $(n \times n)$ matrix $\Theta := (\theta_{k,i})_{1\le i,k\le n}$ is a doubly stochastic matrix: all its elements are non-negative, and all row- and column-wise sums are equal to one.

For the special case of the assumption in Theorem 3.1 the invariant distribution is written explicitly, so is θ_{ki}. In fact, we can compute from $\theta_{ki} = \sum_{\{\pi\in\Pi:\pi(k)=i\}} \theta_\pi$ where

$$\theta_\pi := \lim_{T\to\infty} \frac{1}{T} \int_0^T 1_{\{Y_{\pi(1)} > \cdots > Y_{\pi(n)}\}}(Y(t))\, dt \quad \mathbb{P} - a.s. \tag{3.67}$$

is given by

$$\theta_\pi = \left(\sum_{\pi\in\Pi} \prod_{j=1}^{n-1} (\widetilde{\nu}_j(\pi))^{-1} \right)^{-1} \prod_{j=1}^{n-1} (\widetilde{\nu}_j(\pi))^{-1}, \tag{3.68}$$

where $\widetilde{\nu}_j(\pi)$ is defined in (3.48) for $\pi \in \Pi$. This formula is obtained by A. Banner.

Growth Rate of Local Times [26].

Under the linear growth condition (3.12) the ranked process $(X_{(1)}(\cdot) \ge X_2(\cdot) \ge \cdots \ge X_n(\cdot))$ satisfies (3.41), namely,

$$X_{(k)}(T) = X_{(k)}(0) + \int_0^T (\gamma + g_k + \gamma_{p_t(k)})\, dt + \frac{1}{2}(\Lambda^{k,k+1} - \Lambda^{k-1,k}), \tag{3.69}$$

and hence, for $k = 1, \ldots, n-1$,

$$X_{(k)}(T) - X_{k+1}(T)$$

$$= X_{(k)}(0) - X_{(k+1)}(0) + \int_0^T (g_k + \gamma_{p_t(k)})\, d\,t - \int_0^T (g_{k+1} + \gamma_{p_t(k+1)}\, d\,t \tag{3.70}$$

$$- \frac{1}{2}\big[\Lambda^{k-1,k}(T) + \Lambda^{k+1,k+2}(T)\big) + \widetilde{\sigma}_k B_k(T) - \widetilde{\sigma}_{k+1} B_{k+1}(T)\,.$$

By the Strong Law of Large Numbers for Brownian motions, the ergodic property (3.53), and

$$\lim_{T\to\infty} \frac{X_{(k)}(T) - X_{(k+1)}(T)}{T} = 0\,, \quad a.s., \tag{3.71}$$

we obtain

$$\lim_{T\to\infty} \frac{1}{T}\Big[\frac{\Lambda^{\ell-1,\ell}(T) + \Lambda^{\ell+1,\ell+2}(T)}{2} - \Lambda^{\ell,\ell+1}(T) + \int_0^T (g_{\ell+1} + \gamma_{p_t(\ell+1)})\, d\,t - \int_0^T (g_\ell + \gamma_{p_t(\ell)})\, d\,t\Big] = 0\,, \quad a.s. \tag{3.72}$$

for every $\ell = 1, \ldots, n-1$. Adding up in this equation over $\ell = k, \ldots, n-1$, we obtain

$$\lim_{T\to\infty} \frac{1}{T}\Big[\frac{\Lambda^{k-1,k}(T) - \Lambda^{k,k+1}(T) - \Lambda^{n-1,n}(T)}{2} + \int_0^T (g_n + \gamma_{p_t(n)})\, d\,t - \int_0^T (g_k + \gamma_{p_t(k)})\, d\,t\Big] = 0\,, \quad a.s. \tag{3.73}$$

for every $k = 1, \ldots, n-1$. Adding up now over all these values of k, we obtain

$$\lim_{T\to\infty} \frac{\Lambda^{n-1,n}(T)}{T} = 2 \lim_{T\to\infty} \frac{1}{T} \int_0^T (g_n + \gamma_{p_t(n)}\, d\,t = 2(g_n + \sum_{i=1}^n \gamma_i \theta_{k,i})\,. \tag{3.74}$$

In conjunction with (3.73), we obtain from this

$$\lim_{T\to\infty} \frac{1}{T}\Big[\frac{\Lambda^{k-1,k}(T) - \Lambda^{k,k+1}(T)}{2} - \int_0^T (g_k + \gamma_{p_t(k)})\, d\,t\Big] = 0\,, \quad a.s. \tag{3.75}$$

But now it follows from this and (3.69) that

$$\lim_{T\to\infty} \frac{X_{(k)}(T)}{T} = \gamma\,, \quad a.s. \tag{3.76}$$

and moreover, we obtain

$$\lim_{T\to\infty} \frac{\Lambda^{k,k+1}(T)}{T} = -2\sum_{\ell=1}^{k}(g_\ell + \sum_{i=1}^{n}\theta_{\ell,i}\gamma_i) > 0\,, \quad a.s. \tag{3.77}$$

and

$$\lim_{T\to\infty} \frac{1}{T}\sum_{k=1}^{n} g_k \int_0^T \mathbf{1}_{Q_i^{(k)}}(X(t))\,dt = -\gamma_i\,, \quad a.s.; \quad 1 \le i \le n\,. \tag{3.78}$$

This means

$$\sum_{k=1}^{n} g_k\,\theta_{k,i} = -\gamma_i\,; \quad 1 \le i \le n\,. \tag{3.79}$$

or equivalently,

$$\sum_{k=1}^{n}(g_k + \gamma_i)\theta_{k,i} = 0\,; \quad 1 \le i \le n\,. \tag{3.80}$$

This is a *probabilistic* proof of (3.80) given in [26] under the linear growth condition (3.12). A. Banner also showed it through algebraic computations [7]. Here is another proof of (3.80) under the linear growth condition (3.12). This would be a sanity check of the probability density (3.48). Consider a subset R of $\mathbb{R}^{2n}$ defined by

$$R := \left\{ (g_1, \ldots, g_n, \gamma_1, \ldots, \gamma_n) \in \mathbb{R}^{2n} \;\middle|\; \begin{array}{c} g_1 + \cdots + g_n + \gamma_1 + \cdots + \gamma_n = 0\,, \\ g_1 + \cdots + g_j + \gamma_{\pi(1)} + \cdots + \gamma_{\pi(j)} \neq 0\,, \\ \text{for each } j = 1, \ldots, n-1\,,\ \pi \in \Pi \end{array} \right\}. \tag{3.81}$$

First observe for $\ell = 2, \ldots, n$,

$$\begin{aligned} &\sum_{\{\pi:\pi(\ell-1)=i\}} (g_1 + \cdots + g_{\ell-1} + \gamma_{\pi(1)} + \cdots + \gamma_{\pi(\ell-1)})\theta_\pi + \sum_{\{\pi:\pi(\ell)=i\}} (g_\ell + \gamma_i)\theta_\pi \\ &= \sum_{\{\pi:\pi(\ell)=i\}} (g_1 + \cdots + g_\ell + \gamma_{\pi(1)} + \cdots + \gamma_{\pi(\ell)})\theta_\pi\,. \end{aligned} \tag{3.82}$$

In fact, define a permutation π' from $\pi \in \{\widetilde{\pi} : \widetilde{\pi}(\ell-1) = i\}$ by

$$\pi'(k) := \pi'(k\,;\pi) = \begin{cases} \pi(k)\,, \ k = 1, \ldots, \ell-2, \ell+1, \ldots, n\,, \\ \pi(\ell)\,, \quad k = \ell-1\,, \\ i\,, \quad k = \ell\,, \end{cases} \tag{3.83}$$

which is obtained by exchanging $(\ell-1)$th and ℓth element of $\pi \in \{\widetilde{\pi} : \widetilde{\pi}(\ell-1) = j\,\}$, and also define $M := (\sum_{\pi\in\Pi}\prod_{j=1}^{n-1}\widetilde{\nu}_j(\pi))^{-1}$. Then, the first term of the left-hand of the equation (3.82) is

$$\begin{aligned}
&\sum_{\{\pi:\pi(\ell-1)=i\}} (-\widetilde{\nu}_{\ell-1}(\pi))^{-1} M \prod_{j=1}^{n-1} \widetilde{\nu}_j(\pi) \\
&= M \sum_{\{\pi:\pi(\ell-1)=i\}} (-1)\cdot\widetilde{\nu}_1(\pi)\cdots\widetilde{\nu}_{\ell-2}(\pi)\widetilde{\nu}_{\ell-2}(\pi)\widetilde{\nu}_{\ell}(\pi)\widetilde{\nu}_{\ell+1}(\pi)\cdots\widetilde{\nu}_{n-1}(\pi) \\
&= M \sum_{\{\pi':\pi'(\ell)=i\}} (-1)\cdot\widetilde{\nu}_1(\pi')\cdots\widetilde{\nu}_{\ell-2}(\pi')\widetilde{\nu}_{\ell-2}(\pi')\widetilde{\nu}_{\ell}(\pi')\widetilde{\nu}_{\ell+1}(\pi')\cdots\widetilde{\nu}_{n-1}(\pi')\,.
\end{aligned}$$

This is because on the set $\{\widetilde{\pi}\, : \widetilde{\pi}(\ell-1) = i\}$, (i) for $k = 1, \ldots, \ell-2$,

$$\begin{aligned}
\widetilde{\nu}_k(\pi) &= -(g_1 + \cdots + g_k + \gamma_{\pi(1)} + \cdots + \gamma_{\pi(k)})^{-1} \\
&= -(g_1 + \cdots + g_k + \gamma_{\pi'(1)} + \cdots + \gamma_{\pi'(k)})^{-1} = \widetilde{\nu}_k(\pi')\,,
\end{aligned}$$

(ii) for $k = \ell+1, \ldots, n-1$,

$$\begin{aligned}
&\widetilde{\nu}_k(\pi) = -(g_1 + \cdots + g_k + \gamma_{\pi(1)} + \cdots + \gamma_{\pi(\ell-2)} + \gamma_i + \gamma_{\pi(\ell)} + \gamma_{\pi(\ell+1)} + \cdots + \gamma_{\pi(k)})^{-1} \\
&= -(g_1 + \cdots + g_k + \gamma_{\pi'(1)} + \cdots + \gamma_{\pi'(\ell-2)} + \gamma_{\pi'(\ell)} + \gamma_{\pi'(\ell-1)} + \gamma_{\pi'(\ell+1)} + \cdots + \gamma_{\pi'(k)})^{-1} \\
&= \widetilde{\nu}_k(\pi')\,,
\end{aligned}$$

and (iii) for $k = \ell$,

$$\begin{aligned}
\widetilde{\nu}_\ell(\pi) &= -(g_1 + \cdots + g_\ell + \gamma_{\pi(1)} + \cdots + \gamma_{\pi(\ell-2)} + \gamma_i + \gamma_{\pi(\ell)})^{-1} \\
&= -(g_1 + \cdots + g_\ell + \gamma_{\pi'(1)} + \cdots + \gamma_{\pi'(\ell-2)} + \gamma_{\pi'(\ell)} + \gamma_{\pi'(\ell-1)})^{-1} = \widetilde{\nu}_\ell(\pi')\,,
\end{aligned}$$

and $\{\widetilde{\pi} : \widetilde{\pi}(\ell - 1) = i\} = \{\widetilde{\pi}' : \widetilde{\pi}'(\ell) = i\}$. Then, the left-hand of (3.82) is

$$
\begin{aligned}
&M \sum_{\{\pi':\pi'(\ell)=i\}} (-1) \cdot \widetilde{\nu}_1(\pi') \cdots \widetilde{\nu}_{\ell-2}(\pi') \widetilde{\nu}_\ell(\pi') \widetilde{\nu}_{\ell+1}(\pi') \cdots \widetilde{\nu}_{n-1}(\pi') \\
&\quad + M \sum_{\{\pi:\pi(\ell)=i\}} (g_\ell + \gamma_i) \prod_{j=1}^{n-1} \widetilde{\nu}_j(\pi) \\
&= M \sum_{\{\pi:\pi(\ell)=i\}} \prod_{j=1}^{n-1} \widetilde{\nu}_j(\pi) [(-\widetilde{\nu}_{\ell-1}(\pi))^{-1} + g_\ell + \gamma_i] \\
&= \sum_{\{\pi:\pi(\ell)=i\}} \theta_\pi (g_1 + \cdots + g_{\ell-1} + \gamma_{\pi(1)} + \cdots + \gamma_{\pi(\ell-1)} + g_\ell + \gamma_{\pi(\ell)}),
\end{aligned}
$$

which is the right-hand of (3.82). Now applying (3.82) for $\ell = 2, \ldots, n$, we obtain

$$
\begin{aligned}
&\sum_{k=1}^n (g_k + \gamma_i)\theta_{k,i} = (g_1 + \gamma_i)\theta_{1,i} + (g_2 + \gamma_i)\theta_{2,i} + \sum_{k=3}^n (g_k + \gamma_i)\theta_{k,i} \\
&= \sum_{\{\pi:\pi(1)=i\}} (g_1 + \gamma_{\pi(1)})\theta_\pi + \sum_{\{\pi:\pi(2)=i\}} (g_2 + \gamma_i)\theta_\pi + \sum_{k=3}^n (g_k + \gamma_i)\theta_{k,i} \\
&= \sum_{\{\pi(2)=i\}} (g_1 + g_2 + \gamma_{\pi(1)} + \gamma_{\pi(2)})\theta_\pi + (g_3 + \gamma_i)\theta_{3,i} + \sum_{k=4}^n (g_k + \gamma_i)\theta_{k,i} \\
&= \sum_{\{\pi(2)=i\}} (g_1 + g_2 + \gamma_{\pi(1)} + \gamma_{\pi(2)})\theta_\pi + \sum_{\{\pi:\pi(3)=i\}} (g_3 + \gamma_i)\theta_\pi + \sum_{k=4}^n (g_k + \gamma_i)\theta_{k,i} \\
&= \sum_{\{\pi:\pi(3)=i\}} (g_1 + g_2 + g_3 + \gamma_{\pi(1)} + \gamma_{\pi(2)} + \gamma_{\pi(3)})\theta_\pi + \sum_{k=4}^n (g_k + \gamma_i)\theta_{k,i} \\
&= \cdots = \sum_{\{\pi:\pi(n)=i\}} (g_1 + \cdots + g_n + \gamma_{\pi(1)} + \cdots + \gamma_{\pi(n)})\theta_\pi \\
&= (g_1 + \cdots + g_n + \gamma_1 + \cdots + \gamma_n)\theta_{n,i} = 0,
\end{aligned}
$$

for $i = 1, \ldots, n$ in the above set R. Therefore, under the assumptions (3.39) and (3.12) it is verified that (3.80) holds. Note that (3.80) holds without the assumption (3.12) as we derived from the Strong Law of Large Numbers. $\square$

Invariant Distribution of Market Share

Since the process $X(\cdot)$ in (3.38) represents the log capitalization of each stock,

$$
(3.84) \qquad \left(\mathfrak{m}_1(t) := \frac{e^{X_{(1)}(t)}}{e^{X_1(t)} + \cdots + e^{X_n(t)}}, \ldots, \mathfrak{m}_n(t) := \frac{e^{X_{(n)}(t)}}{e^{X_1(t)} + \cdots + e^{X_n(t)}}, 0 \le t < \infty \right)
$$

are the market shares in the financial Atlas model, i.e., $\mathfrak{m}_k$ is the kth largest company's share in the market. The stationary distribution of market shares are computed under the assumption of Theorem 3.1:

$$\begin{aligned} &\wp(m_1, \ldots, m_{n-1}) \\ &= \sum_{\pi \in \Pi} \theta_\pi \frac{\tilde{\nu}_1(\pi) \cdots \tilde{\nu}_{n-1}(\pi)}{m_1^{\tilde{\nu}_1(\pi)+1} \cdot m_2^{\tilde{\nu}_2(\pi)-\tilde{\nu}_1(\pi)+1} \cdots m_{n-1}^{\tilde{\nu}_{n-1}(\pi)-\tilde{\nu}_{n-2}(\pi)+1} m_n^{-\tilde{\nu}_{n-1}(\pi)+1}}, \\ &\qquad 0 < m_n \le m_{n-1} \le \ldots \le m_1 < 1, \quad m_n = 1 - m_1 - \ldots - m_{n-1}. \end{aligned} \tag{3.85}$$

Remark 3.7. As a concluding remark, let us discuss some open problems in the Atlas model.

- (Portfolio optimizations) Under a class of equity market models with some ergodic properties, Cover [10] & Jamshidian [25] introduce the so-called *universal portfolio*. The authors consider the *target portfolio*, which maximizes the self-financing portfolio, as a benchmark for evaluating the long-term performance of constant-proportion portfolios and claim the performance of universal portfolio is optimal in some asymptotic sense. Karatzas [27] compares the performance of the universal portfolio vis-à-vis the target performance of the target portfolio and the growth optimal portfolio under the Atlas model. The beauty of universal portfolio is that it is unnecessary to know all the model parameters. One of the interesting questions is how much a portfolio manager can improve the performance comparative to the universal portfolio with the additional information about the model parameters.

- (No-triple-collision) Propositions 1.6 in Section 1.6 and 3.2 in Section 3.2 are sufficient conditions for no-triple collision of the process $X(\cdot)$ with piecewise constant diffusion coëfficients. Is there an easily verifiable necessary and sufficient criterion for no-triple-collision?

- (Circulation time) The average occupation time formula (3.56) shows that the process stays in the region $Q_i^{(k)}$ for the amount $\theta_{k,i}$ of time on average, for $1 \le i, k \le n$. The $(n \times n)$ matrix $\Theta := (\theta_{k,i})$ is the doubly stochastic matrix. Does the matrix Θ contain some information about how fast the process switches from one region to another? Or about how long does it take the top-ranked one go down to the bottom rank, then come up back to the top?

- (Statistical estimation) Statistical estimation of model parameters is another research topic.

The model is similar to the self-exciting threshold-type discrete-time model studied by Tong & Lim [58]. Fernholz [12] and Karatzas [26] propose some estimators of local times and model parameters.

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