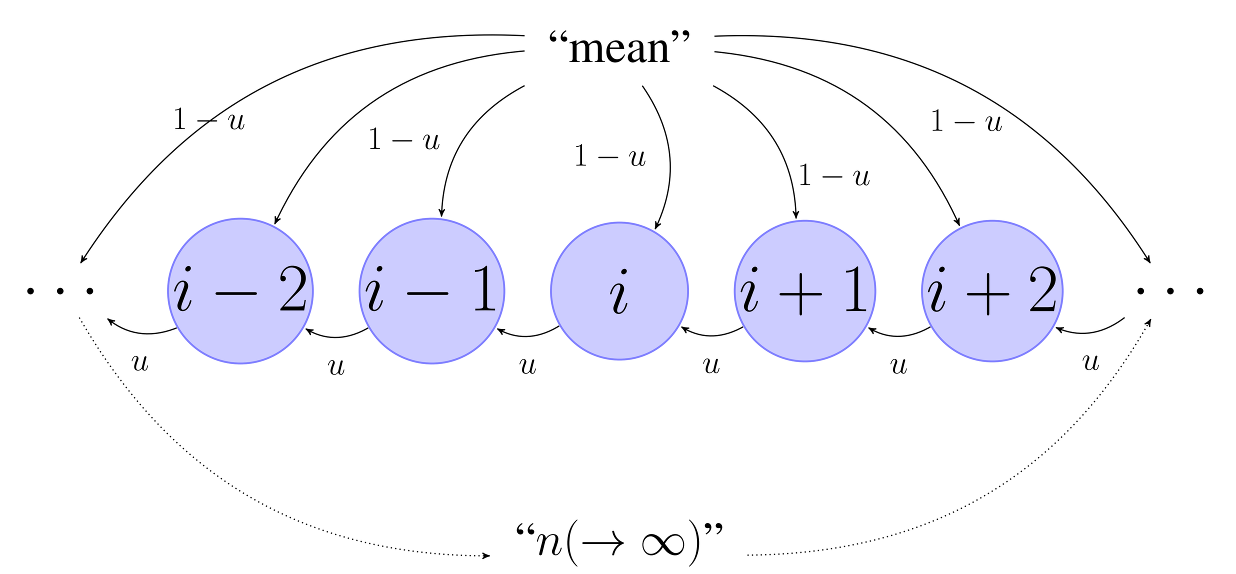

Detecting Mean-field

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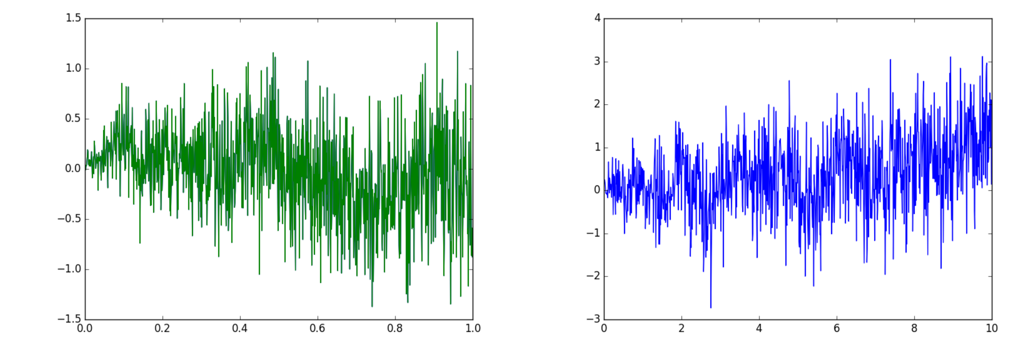
based on joint work [DFI] with Nils Detering and Jean-Pierre Fouque



Simulated sample paths of pair: $X_\cdot^{(1)}$ (green) $0 \le t \le 1$, $\widetilde{X}_\cdot^{(1)}$ (blue) $0 \le t \le 10$ with $u = 1$.

1. Questions and Motivation

Let us consider a linear torus directed graph (or directed network) of vertices $\{1, \dots, n\}$ in the sense that each node i in the network connects only with its neighboring vertex $i+1$ for $i = 1, \dots, n-1$, and the boundary vertex n connects with vertex 1.

On some probability space, based on this torus graph let us consider the simple Ornstein-Uhlenbeck type system (or a Gaussian cascade)

$$\begin{aligned} \mathrm{d}X_{t,i} &= (X_{t,i+1} - X_{t,i})\mathrm{d}t + \mathrm{d}W_{t,i}\,; \quad t \ge 0\,, \quad i = 1, \dots, n-1\,, \\ \mathrm{d}X_{t,n} &= (X_{t,1} - X_{t,n})\mathrm{d}t + \mathrm{d}W_{t,n} \end{aligned} \tag{1}$$

with initial independent and identically distributed random variables $X_{0,i}$, independent of standard Brownian motions $(W_{\cdot,i})$, $1 \le i \le n$.

For comparison, on the same probability space, we consider a typical mean-field interacting system where each particle is attracted towards the mean, defined by

$$\mathrm{d}X_{t,i} = \Big(\frac{1}{n}\sum_{j=1}^{n} X_{t,j} - X_{t,i}\Big)\mathrm{d}t + \mathrm{d}W_{t,i}\,; \quad t \ge 0\,, \quad i = 1, \dots, n\,. \tag{2}$$

The particle $X_{\cdot,i}$ at node i is *directly* attracted towards the mean $(X_{\cdot,1} + \cdots + X_{\cdot,n})/n$ of the system.

Questions.
Q1. What is the essential difference between the system (1) and (2) for large $n \to \infty$?
Q2. Can we detect the type of interaction from the particle behavior at one node?

Motivation.
- Effects of graph (network) structure: mean-field type and local directed chain dependence.

2. An extension of McKean-Vlasov stochastic equation with constraints

Let us introduce a mixed system of linear equations:

$$\begin{aligned} \mathrm{d}X_{t,i} &= \Big(u \cdot X_{t,i+1} + (1-u)\cdot \frac{1}{n}\sum_{j=1}^{n} X_{t,j} - X_{t,i}\Big)\mathrm{d}t + \mathrm{d}W_{t,i}\,, \\ \mathrm{d}X_{t,n} &= \Big(u \cdot X_{t,1} + (1-u)\cdot \frac{1}{n}\sum_{j=1}^{n} X_{t,j} - X_{t,n}\Big)\mathrm{d}t + \mathrm{d}W_{t,n} \end{aligned} \tag{3}$$

for $t \ge 0$, $i = 1, \dots, n-1$ with the initial $X_{0,i}$, $1 \le i \le n$, and for fixed $u \in [0,1]$. (3) contains both (1) $u = 1$ and (2) $u = 0$.

- A bit more generally, on a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t), \mathbb{P})$, given a constant $u \in [0,1]$ and a functional $b : [0,\infty) \times \mathbb{R} \times \mathcal{M}(\mathbb{R}) \to \mathbb{R}$, let us consider a non-linear diffusion pair $(X_t^{(u)}, \widetilde{X}_t^{(u)})$, $t \ge 0$, described by the stochastic differential equation of McKean-Vlasov type:

$$\mathrm{d}X_t^{(u)} = b(t, X_t^{(u)}, F_t^{(u)})\,\mathrm{d}t + \mathrm{d}B_t\,; \quad t \ge 0\,, \tag{4}$$

driven by a BM $(B_t, t \ge 0)$, where $F_\cdot^{(u)}$ is the weighted p.m.

$$F_t^{(u)}(\cdot) := u \cdot \delta_{\widetilde{X}_t^{(u)}}(\cdot) + (1-u)\cdot \mathcal{L}_{X_t^{(u)}}(\cdot) \tag{5}$$

of the Dirac mass $\delta_{\widetilde{X}_t^{(u)}}(\cdot)$ of $\widetilde{X}_t^{(u)}$ and law $\mathcal{L}_{X_t^{(u)}}$ of $X_t^{(u)}$. We shall assume that the law of $X_\cdot^{(u)}$ is identical to that of $\widetilde{X}_\cdot^{(u)}$, and $\widetilde{X}_\cdot^{(u)}$ is independent of the Brownian motion, i.e.,

$$\mathrm{Law}((X_t^{(u)}, t \ge 0)) \equiv \mathrm{Law}((\widetilde{X}_t^{(u)}, t \ge 0)) \quad \text{and} \tag{6}$$

$$\boldsymbol{\sigma}(\widetilde{X}_t^{(u)}, t \ge 0) \perp\!\!\!\perp \boldsymbol{\sigma}(B_t, t \ge 0)\,.$$

Let us also assume that the Brownian motion $B_\cdot$ is independent of the initial value $(X_0^{(u)}, \widetilde{X}_0^{(u)})$.

Proposition. *Suppose that $b(\cdot)$ is Lipschitz and is at most linear growth. Then, for each $u \in [0,1]$ there exists a weak solution $(\Omega, \mathcal{F}, (\mathcal{F}_t), \mathbb{P})$, $(X_\cdot^{(u)}, \widetilde{X}_\cdot^{(u)}, B_\cdot)$ to the infinite-dimensional McKean-Vlasov equation (4) with (5)-(6). This solution is unique in law.*

Let us take a linear functional $b(t, x, \mu) := -\int_{\mathbb{R}}(x - y)\mu(\mathrm{d}y)$ for $t \ge 0$, $x \in \mathbb{R}$, $\mu \in \mathcal{M}(\mathbb{R})$ of mean-reverting type. (4) is reduced to

$$\mathrm{d}X_t^{(u)} = -\Big(u\,(X_t^{(u)} - \widetilde{X}_t^{(u)}) + (1-u)(X_t^{u} - \mathbb{E}[X_t^{(u)}])\Big)\,\mathrm{d}t + \mathrm{d}B_t\,. \tag{7}$$

- As $n \to \infty$, the first two components $(X_{\cdot,1}, X_{\cdot,2})$ of (3) converges weakly to $(X_\cdot^{(u)}, \widetilde{X}_\cdot^{(u)})$ in (7). Following the martingale method [O], one can show that the joint distribution and marginal distribution of $(X_\cdot^{(u)}, \widetilde{X}_\cdot^{(u)})$ satisfies an integral equation. The details, variants and fluctuation results are found in [DFI]. This answers the question Q1.

3. Detecting presence of mean-field

Assume $X_0^{(u)} \equiv 0 \equiv \widetilde{X}_0^{(u)}$. Let us consider the following problem of a single observer: The observer only observes $X_t := X_t^{(u)}$, $t \ge 0$ but does neither know the value $u \in [0,1]$ nor $\widetilde{X}_t^{(u)}$, $t \ge 0$ in (7).

The value u in (7) indicates how much $X_\cdot^{(u)}$ is attracted towards the neighborhood $\widetilde{X}_\cdot^{(u)}$ and $(1-u)$ indicates how much it is attracted towards the average $\mathbb{E}[X_t^{(u)}]$ $(= 0)$. In this context, Q2 is rephrased as

Q2′. Only given the filtration $\mathcal{F}_t^X := \sigma(X_s^{(u)}, 0 \le s \le t)$, $t \ge 0$, can the observer detect the value $u \in [0,1]$?

Yes, after observing for a sufficiently long time! (7) with (6) is solvable explicitly, and a method-of moments estimator is consistent :

$$\lim_{T \to \infty} \widehat{u}_M = u \quad a.s.\,,$$

where

$$\widehat{u}_M := \Big[1 - \Big(\frac{2}{T}\int_0^T X_t^2 \mathrm{d}t\Big)^{-1/2}\Big]^{1/2}. \tag{8}$$

A modified version

$$\begin{aligned} \widehat{u}_m &:= \Big(\int_0^T X_t^2 \mathrm{d}t\Big)^{-1} \cdot \Big(\int_0^T X_t^2 \mathrm{d}t + \int_0^T X_t \mathrm{d}X_t\Big) \\ &= 1 - \Big(2\int_0^T X_t^2 \mathrm{d}t\Big)^{-1}\big(T - X_T^2\big)\,. \end{aligned}$$

of the conditional maximum likelihood estimator

$$\widehat{u} := \Big(\int_0^T \mathbb{E}[\widetilde{X}_t^2 | \mathcal{F}_T^X]\mathrm{d}t\Big)^{-1} \cdot \mathbb{E}\Big[\int_0^T X_t \widetilde{X}_t \mathrm{d}t + \int_0^T \widetilde{X}_t \mathrm{d}X_t \,\Big|\, \mathcal{F}_T^X\Big]$$

underestimates the value, i.e., $\lim_{T \to \infty} \widehat{u}_m = 1 - \sqrt{1 - u^2} \le u \in [0,1]$. The detailed study of $\widehat{u}$ and filtering problem is an ongoing research.

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