

# Stochastic Integral Equations for Walsh Semimartingales

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## 1. Questions

The WALSH Brownian motion introduced in [W] is a two-dimensional analogue of the skew Brownian motion (e.g., [HS]) with a spinning probability measure  $\boldsymbol{\mu}$  on  $\mathcal{B}(\mathfrak{S})$ , where  $\mathfrak{S}$  is the unit circumference

$$\mathfrak{S} := \{(x_1, x_2) \in \mathbb{R}^2 : x_1^2 + x_2^2 = 1\},$$

instead of Bernoulli probability on  $\mathcal{B}(\{-1, 1\})$ .

There are several constructions of WALSH Brownian motions. Here on a filtered probability space  $(\Omega, \mathcal{F}, \mathbb{P}), \mathbb{F} = (\mathcal{F}_t)_{0 \leq t < \infty}$  let us consider the following system of *degenerate* stochastic integral equations for  $X(t) := (X_1(t), X_2(t))$ ,  $0 \leq t < \infty$ :

$$X_i(\cdot) = x_i + \int_0^\cdot f_i(X(t)) dS(t) + \gamma_i L^S(\cdot), \quad (1)$$

for  $i = 1, 2$ , driven by its radial part  $\|X(\cdot)\| = S(\cdot)$ , where  $f(x) := (f_1(x), f_2(x)) = x / \|x\|$  for  $x \in \mathbb{R}^2 \setminus \{\mathbf{0}\}$ ,  $f(\mathbf{0}) := 0$ ,  $S(\cdot)$  is the SKOROKHOD reflection (or folding)

$$S(\cdot) := U(\cdot) + \max_{0 \leq s \leq \cdot} (-U(s))^+,$$

of a real-valued, continuous semimartingale  $U(\cdot)$ ,

$$\gamma_i := \int_{\mathfrak{S}} f_i(z) \boldsymbol{\mu}(dz), \quad i = 1, 2,$$

and  $L^\Xi(\cdot)$  is the local time accumulated at the origin during the time-interval  $[0, \cdot]$  by a generic one-dimensional continuous semimartingale  $\Xi(\cdot)$ :

$$L^\Xi(\cdot) := \lim_{\varepsilon \downarrow 0} \frac{1}{2\varepsilon} \int_0^\cdot \mathbf{1}_{\{0 \leq \Xi(t) < \varepsilon\}} d\langle \Xi \rangle(t).$$

Assume for concreteness  $\text{Leb}(\{t : S(t) = 0\}) = 0$  and  $L^S(\infty) > 0$  a.s. Here (1) can be seen as the two-dimensional analogue of HARRISON-SHEPP equation [HS] for the skew Brownian motion.

**Q.** Can we solve (1)? If so, what can we say about uniqueness and other properties?

## 2. Change-of-variable formula

**A. Yes**, by *folding and unfolding* the semimartingales, one can construct a planar process  $X(\cdot)$  that satisfies (1) and that for every  $A \in \mathcal{B}([0, 2\pi))$ ,

$$L^{R^A}(\cdot) \equiv \mathbf{v}(A) L^{\|X\|}(\cdot), \quad (2)$$

where  $R^A(\cdot) := \|X(\cdot)\| \cdot \mathbf{1}_A(\arg(X(\cdot)))$  is the “thinned” semimartingale, and  $\mathbf{v}(d\theta) := \boldsymbol{\mu}(dz)$ ,  $z \in \mathfrak{S}$ ,  $\theta \in [0, 2\pi)$ .

It also satisfies the FREIDLIN-SHEU like formula

$$dg(X(\cdot)) = \left( \int_0^{2\pi} g'_\theta(0+) \mathbf{v}(d\theta) \right) dL^S(\cdot) + \mathbf{1}_{\{X(\cdot) \neq \mathbf{0}\}} \left( G'(X(\cdot)) dU(\cdot) + \frac{1}{2} G''(X(\cdot)) d\langle U \rangle(\cdot) \right)$$

for every function  $g : \mathbb{R}^2 \rightarrow \mathbb{R}$  that is continuous in the topology induced by the tree-metric, and for every  $\theta \in [0, 2\pi)$  the function  $r \mapsto g_\theta(r) := g(r, \theta)$  is  $C^2((0, \infty))$  with  $G'(x) := g'_\theta(r)$ ,  $G''(x) := g''_\theta(r)$  for  $x = (r, \theta)$ .

In particular, if  $G''(\cdot) \equiv 0$ ,  $\int_0^{2\pi} g'_\theta(0+) \mathbf{v}(d\theta) \equiv 0$ , and if  $U(\cdot)$  is a continuous local martingale, then so is  $g(X(\cdot))$ . This leads to *martingale characterizations*; For example, we have the following

**Proposition.** When  $U(\cdot)$  is a standard Brownian motion, the weak solution to the system (1) of equations that satisfies (2) is unique in the sense of probability distribution, and is the WALSH Brownian motion with the spinning measure  $\boldsymbol{\mu}$ .

## 3. Angular dependence and more

By time-change we may consider the corresponding submartingale problems for angular dependent infinitesimal generators. More elaborate, full discussion on diffusions of this type and solvability of (1) is in [IKPY] and references therein.

## References

- [HS] Harrison, J.M. & Shepp, L.A. (1981) On skew Brownian motion. *Annals of Probability* **9**, 309-313.
- [IKPY] ICHIBA, T., KARATZAS, I., PROKAJ, V. & YAN, M. (2015) Stochastic Integral Equations for Walsh Semimartingales. arXiv 1505.02504.
- [W] Walsh, J.B. (1978) A diffusion with a discontinuous local time. *Astérisque* **52-53**, 37-45.