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STRONG/WEAK SOLUTIONS OF 2D DIFFUSIONS WITH RANK-BASED CHARACTERISTICS

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Abstract

We consider a class of two-dimensional diffusions interacting through rank-based characteristics. In terms of the (possibly degenerate) rank-based diffusion matrix we categorize the strength and weakness of solutions to the corresponding stochastic differential equations. Planar Brownian motions generated by some combinatorial considerations have different degrees of information. This poster is based on [FIKP].

1. Introduction

Q. Given a stochastic differential equation, what determines whether it admits a pathwise-unique solution or not?

A. Intuitively, if the equation specifies clearly enough the corresponding dynamics, it admits a strong solution; otherwise, it does not. □

For example, recall the one-dimensional TANAKA equation

$$\mathrm{d}X(t) = \mathrm{sgn}(X(t))\mathrm{d}B(t)\,; \quad 0 \le t < \infty, \tag{1}$$

where $B(\cdot)$ is a Brownian motion and $\mathrm{sgn}(x) := \mathbf{1}_{\{x>0\}} - \mathbf{1}_{\{x\le 0\}}$. It has *a weak solution unique in distribution, but does not admit a strong solution* (e.g., see section 5.3 of [KS]).

On the other hand, the one-dimensional perturbed TANAKA equation

$$\mathrm{d}X(t) = \mathrm{sgn}(X(t))\mathrm{d}B_1(t) + \mathrm{d}B_2(t)\,; \quad 0 \le t < \infty, \tag{2}$$

admits a strong solution, where $(B_1(\cdot), B_2(\cdot))$ is a planar BM (see [P]). In (1) the diffusion coefficient 1 (for $x>0$) and -1 (for $x \le 0$) points in exactly opposite directions, which creates some freedom (or ambiguity) of choosing positive/negative excursions of BM for the paths of $X(\cdot)$. In (2) the additional noise diminish this ambiguity, and (2) admits a strong solution. We discuss some two-dimensional analogue of this phenomenon through diffusions with rank-based characteristics.

2. 2D Rank-based Diffusions

Given nonnegative constants ρ, σ with $\rho^2 + \sigma^2 = 1$, let us consider the following two-dimensional diffusion $\boldsymbol{X}(\cdot) := (X_1(\cdot), X_2(\cdot))'$ with

$$\mathrm{d}\boldsymbol{X}(t) = \Sigma(X_1(t), X_2(t))\,\mathrm{d}\boldsymbol{B}(t)\,, \tag{3}$$

where $\boldsymbol{B}(\cdot) := (B_1(\cdot), B_2(\cdot))'$ is a planar BM, and $\Sigma(x_1,x_2) := \Sigma_+ \mathbf{1}_{\{x_1>x_2\}} + \Sigma_- \mathbf{1}_{\{x_1 \le x_2\}}$ with $\Sigma(x_1,x_2)\Sigma(x_1,x_2)' = \mathscr{A}(x_1,x_2)$,

$$\mathscr{A}(x_1,x_2) := \begin{pmatrix} \rho^2 \mathbf{1}_{\{x_1>x_2\}} + \sigma^2 \mathbf{1}_{\{x_1 \le x_2\}} & 0 \\ 0 & \rho^2 \mathbf{1}_{\{x_1 \le x_2\}} + \sigma^2 \mathbf{1}_{\{x_1 > x_2\}} \end{pmatrix}.$$

Note that with the notation of $\boldsymbol{e}_1 = (1,0)'$, $\boldsymbol{e}_2 = (0,1)'$, the sign of $Y(\cdot) := (\boldsymbol{e}_1 - \boldsymbol{e}_2)\boldsymbol{X}(\cdot) = X_1(\cdot) - X_2(\cdot)$ determines the rank of $(X_1(\cdot), X_2(\cdot))$, i.e., $X_1(\cdot)$ is bigger than $X_2(\cdot)$ or not.

3. Criterion of Weakness/Strength

Theorem *The (unique in distribution) weak solution of the system (3) fails to be strong, if and only if*

$$(\boldsymbol{e}_1 - \boldsymbol{e}_2)'\Sigma_- = -(\boldsymbol{e}_1 - \boldsymbol{e}_2)'\Sigma_+\,. \tag{4}$$

Proof is in [FIKP]. Under the condition (4), the diffusion vectors $(\boldsymbol{e}_1 - \boldsymbol{e}_2)'\Sigma_+$ (for the half-plane $\{x_1 > x_2\}$) and $(\boldsymbol{e}_1 - \boldsymbol{e}_2)'\Sigma_-$ (for the half-plane $\{x_1 \le x_2\}$) of $Y(\cdot) = (\boldsymbol{e}_1 - \boldsymbol{e}_2)\boldsymbol{X}(\cdot)$ point in *exactly opposite directions*. As in TANAKA equation, it becomes impossible for the planar diffusion $\mathbf{X}(\cdot)$ to "escape strongly from the diagonal $\{x_1 = x_2\}$."

The matrix $\mathscr{A}(x_1,x_2)$ has a total of 64 square roots of the form $\Sigma_1 \mathbf{1}_{\{x_1>x_2\}} + \Sigma_2 \mathbf{1}_{\{x_1 \le x_2\}}$ with

$$\Sigma_1 \in \left\{ \begin{pmatrix} \pm\rho & 0 \\ 0 & \pm\sigma \end{pmatrix}, \begin{pmatrix} 0 & \pm\rho \\ \pm\sigma & 0 \end{pmatrix} \right\},$$

$$\Sigma_2 \in \left\{ \begin{pmatrix} \pm\sigma & 0 \\ 0 & \pm\rho \end{pmatrix}, \begin{pmatrix} 0 & \pm\sigma \\ \pm\rho & 0 \end{pmatrix} \right\}.$$

With Theorem it can be verified that among these, 48 lead to strongly solvable systems in the isotropic ($\rho = \sigma = 1/\sqrt{2}$) or degenerate ($\rho\,\sigma = 0$) cases, whereas 56 choices lead to strongly solvable systems in all other cases.

4. Planar BMs and Filtrations

Along with the same line of analysis let us introduce two planar Brownian motions $(V_1(\cdot), V_2(\cdot))'$ and $(W_1(\cdot), W_2(\cdot))'$:

$$V_1(\cdot) := \int_0^{\cdot} \mathbf{1}_{\{Y(s)>0\}}\,\mathrm{d}B_1(s) + \int_0^{\cdot} \mathbf{1}_{\{Y(s)\le 0\}}\,\mathrm{d}B_2(s)\,,$$

$$V_2(\cdot) := \int_0^{\cdot} \mathbf{1}_{\{Y(s)\le 0\}}\,\mathrm{d}B_1(s) + \int_0^{\cdot} \mathbf{1}_{\{Y(s)>0\}}\,\mathrm{d}B_2(s)\,,$$

$$W_1(\cdot) := \int_0^{\cdot} \mathbf{1}_{\{Y(s)>0\}}\,\mathrm{d}B_1(s) - \int_0^{\cdot} \mathbf{1}_{\{Y(s)\le 0\}}\,\mathrm{d}B_2(s)\,,$$

$$W_2(\cdot) := \int_0^{\cdot} \mathbf{1}_{\{Y(s)\le 0\}}\,\mathrm{d}B_1(s) - \int_0^{\cdot} \mathbf{1}_{\{Y(s)>0\}}\,\mathrm{d}B_2(s)\,.$$

In general, the filtrations $\mathfrak{F}^{(V_1,V_2)}(\cdot)$ and $\mathfrak{F}^{(W_1,W_2)}(\cdot)$ contain different information. For example, if $\Sigma_+ = \mathrm{diag}(\rho,\sigma)$ and $\Sigma_- = \mathrm{diag}(\sigma,\rho)$,

$$\mathfrak{F}^{(V_1,V_2)}(\cdot) \subsetneq \mathfrak{F}^{(W_1,W_2)}(\cdot) = \mathfrak{F}^{(B_1,B_2)}(\cdot) = \mathfrak{F}^{(X_1,X_2)}(\cdot)\,.$$

5. Further Studies

More elaborate, full discussion on diffusion of this type is in [FIKP] and references therein. For some new results in three or higher dimensional case see [IKS].

References

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